Studies on Ultraquasi-pseudometrics and orderings

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A Dissertation presented for the degree of
Master of Science

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September 2017

Contents

Abstract

Ultraquasi-pseudometric spaces even though quite simple in concept, as it is easily obtained by altering the usual triangle inequality property, still yield interesting results. Indeed, a natural question that should arise is how does switching to the strong triangle inequality affect some of the results we already know about quasi-pseudometrics. On some points we get similar results to those of when we have the standard triangle inequality, but the general observation is that dealing with the strong triangle inequality is easier. Of course, there are results that cannot be obtained without the "ultra-property".

A fast rundown on those effects is then deemed necessary to begin with. Also, since we cannot go through every single result on quasi-pseudometrics we then need to localize our observations, that is why in the first part we restrained our observations on the results from Gaba and Künzi about splitting metrics [3].

Also, we will see some particular algorithms and connections to the bicompletion, joincompact ultraquasi-metric spaces and the old construction of a total order by Herrlich [6].

Chapter 1

Introduction

As roughly outlined by the abstract, for this thesis, at first we are going to revise some recent results established about quasi-pseudometrics in [3]. However in doing so, we are going to make the change of using ultraquasi-metrics instead of quasi-metrics and see how that affects the results. That is, we build upon the results from that paper by only using the strong triangle inequality throughout. Compared to [3], this chapter has been written so that people who have not read through that paper or have little to no experience about ultra-quasimetrics could pick and understand the contents at a very fast pace. So, proofs may seem to be a bit too detailed but that is occasionally made part of the studies, as suggested by the title.

Next we are going to have a look at our main result, which enables us to say that the concept of maximally m-produced partial orders and total orders are basically the same in the ultra spaces.

In this part too, we are going to present a simple algorithm which enables us to construct a strictly decreasing chain of splitting ultraquasi-metrics and then we will have a look at a simple example illustrating some concepts that we described on a finite set. The algorithm too is going to be illustrated with this example.

Next we will have a look at some particular alternative results which are specifically applicable to compact ultraquasi-metric spaces. We will look at a particular method of proving that given a compact ultra-metric space, we can always form a totally ordered compact ultra-metric space. Of course, this ordered space is not deemed to be unique. During this we will have a look at a generalization of a result concerning the range of an ultrametric map from [5].

Then we will have a review of Herrlich's [6] famous process of construction of a totally ordered ultrametric space. Basically, the construction starts off by dividing the space into a partition; and then by induction, each of the

element of that partition is divided so that we obtain a new partition but with even more elements and with smaller size, and so on.
An outline of the construction is given in the general topology book by Engelking [1].

Chapter 2

From Quasi-pseudometric to Ultraquasi-pseudometric

In this chapter, as we said, we are going to revise some results on quasipseudometrics in [3] but with that we are going to make a simple modification of using only the strong triangle inequality throughout.

2.1 Basic Definitions

Throughout, $\mathbb{R}_+$ will denote the set of real number greater or equal to zero. By analogy, we also have $\mathbb{Z}_+ = \mathbb{N}$ the set of natural numbers and so on. A star superscription will mean that we exclude zero, as for example $\mathbb{N}^*$ will denote the set of natural numbers excluding zero.

Definition 2.1.1. For a set X, an *ultraquasi-pseudometric* defined over X is a function $d : X \times X \to \mathbb{R}_+$ such that,

(a) $d(x, x) = 0$ for $x \in X$,

(b) $d(x, y) \leq \max(d(x, z), d(z, y)) = d(x, z) \vee d(z, y)$ for $x, y, z \in X$.

An ultraquasi-pseudometric space is then a pair (X, d) such that X is a set and d is an ultraquasi-pseudometric defined over X.

For (X, d) an ultraquasi-pseudometric space, we have the following naming.

- If we have the additional condition,

$$d(x, y) = 0 = d(y, x) \quad \Longrightarrow \quad x = y, \tag{2.1}$$

then d is called an *ultraquasi-metric* and (X, d) is called an ultraquasi-metric space.

- If d satisfies,

$$d(x,\, y) = d(y,\, x), \quad x,\, y \in X, \tag{2.2}$$

then d is called a *ultra-pseudometric* and $(X,\, d)$ is called an ultra-pseudometric space.

- If we both have (2.1) and (2.2), then d is called an ultrametric and $(X,\, d)$ is an ultrametric space.

Notice that if d is an ultraquasi-pseudometric (resp. ultraquasi-metric, ultra-pseudometric, metric) over a set X, then d is a quasi-pseudometric (resp. quasi-metric, pseudometric, ultrametric) over X.

As with quasi-pseudometrics, for an ultraquasi-pseudometric d defined over a set X, the function d^{-1} defined over $X \times X$ by

$$d^{-1}(x,\, y) = d(y,\, x), \qquad x,\, y \in X. \tag{2.3}$$

is also an ultraquasi-pseudometric defined over X, the *conjugate ultraquasi-pseudometric* of d.

For an ultraquasi-pseudometric space $(X,\, d)$, we can also define what is called the *symmetrization* of d by

$$d^s = \max(d,\, d^{-1}) = d \vee d^{-1}. \tag{2.4}$$

Remark 2.1.2.

- The symmetrization d^s of d is an ultra-pseudometric. Indeed, for $x,\, y \in X$,

$$d^s(x,\, y) = \max(d(x,\, y),\, d^{-1}(x,\, y)) = \max(d^{-1}(y,\, x),\, d(y,\, x)) = d^s(y,\, x).$$

- If l and p are ultraquasi-pseudometrics defined such that $l \leq p$, then $l^s \leq p^s$.

Remark 2.1.3. For an ultraquasi-pseudometric space $(X,\, d)$, let us define the binary relation $\leq_d = \{(x,\, y) \in X \times X \,|\, d(x,\, y) = 0\}$. Hence, we have $x \leq_d y$ if and only if $(x,\, y) \in \leq_d$ if and only if $d(x,\, y) = 0$. The binary relation $\leq_d$ is called the *specialization preorder of* d. As its name states, it is a preorder: it is reflexive by (a) and transitive by (b).

Notice that if l and p are ultraquasi-pseudometrics defined on X such that $l \leq p$, then by passing to specialization preorders, we have $\leq_l \supseteq \leq_p$.

Proposition 2.1.4. *The ultraquasi-pseudometric d is an ultraquasi-metric if and only if d^s is an ultrametric on X if and only if $\leq_d$ is a partial order on X.*

Proof. Indeed, suppose d is an ultraquasi-metric. Then, for $x,\, y \in X$, if $d^s(x,\, y) = d^s(x,\, y) = 0$ then $d(x,\, y) = d^{-1}(x,\, y) = d(y,\, x) = d^{-1}(y,\, x) = 0$. Hence, since d is an ultraquasi-metric and $d(x,\, y) = d(y,\, x) = 0$, we have $x = y$.
Conversely, suppose that d^s is an ultrametric. Then for $x,\, y \in X$, if $d(x,\, y) = d(y,\, x) = 0$, then $d(x,\, y) = d^{-1}(x,\, y) = 0$. Therefore,

$$d^s(x,\, y) = \max(d(x,\, y),\, d^{-1}(x,\, y)) = 0.$$

Hence, since d^s is an ultrametric, $d^s(x,\, y) = 0$ implies that $x = y$.
As for the other equivalence; for $(x,\, y) \in X \times X$, we notice that

$$d(x,\, y) = d(y,\, x) = 0 \Longleftrightarrow x \leq_d y \text{ and } y \leq_d x.$$

This clearly shows that d is a ultraquasi-metric if and only if $\leq_d$ is a partial order. $\square$

Given an ultraquasi-pseudometric space $(X,\, d)$, for $x \in X$ and $\epsilon > 0$, we define

$$B_d(x, \epsilon) = \{y \in X \,|\, d(x,\, y) < \epsilon\},$$

the *open ball centred on x of radius ϵ*. Similarly, we define a *closed ball centred on x and of radius ϵ* as the set,

$$\bar{B}_d(x,\, \epsilon) = \{y \in X \,|\, d(x,\, y) \leq \epsilon\}.$$

For $x \in X$, the collection

$$\mathcal{B}_d(x) = \{B_d(x,\, \epsilon) \,|\, \epsilon > 0\},$$

of *all open ball centred on x* defines a base of neighbourhoods at x. This gives us then a topology $\tau(d)$ on X. More explicitly, the collection of *all open balls* on X corresponds to a basis for that topology. The topology $\tau(d)$ is called the *topology induced* by d on X.
Given a closed ball $A = \bar{B}_d(x, \epsilon) \subseteq X$, this subset is closed with respect to $\tau(d^{-1})$. Indeed, $A^c = \{y \in X \,|\, d^{-1}(y,\, x) > \epsilon\}$. Let $y \in A^c$, set $\delta = d^{-1}(y,\, x)$ and denote $C = B_{d^{-1}}(y,\, \delta - \epsilon)$. Therefore, for $z \in C$, we have $d^{-1}(y,\, z) < \delta - \epsilon < \delta = d(y,\, x)$. On the other hand by the strong triangle inequality, we have $d^{-1}(y,\, x) \leq d^{-1}(y,\, z) \vee d^{-1}(z,\, x)$. Hence $d^{-1}(y,\, x) \leq d^{-1}(z,\, x)$. Therefore from the assumption that $y \in A^c$, we then have $d^{-1}(z,\, x) > \epsilon$, so $z \in A^c$ too. Therefore $C = B_{d^{-1}}(y,\, \delta - \epsilon) \subseteq A^c$ and A^c is open with respect to $\tau(d^{-1})$.

Definition 2.1.5. Given a metric space (X, m), an ultraquasi-pseudometric d on X is said *m-splitting* if $d^s = m$.

Definition 2.1.6. If $\leq$ is a partial order on a metric space (X, m), the triple $(X, m, \leq)$ is called a *partially ordered metric space.*

Definition 2.1.7. Let (X, m) be a metric space and d be an ultraquasi-pseudometric on X.

1. If the triple $(X, m, \leq) = (X, m, \leq_d)$, where d is m-splitting, then we say that the partially ordered metric space is *produced* by d (notice here that d has to be an ultraquasi-metric).
   And in this case, the partial order $\leq$ is said to be *m-produced* by d.
2. If d is m-splitting and has the property such that for any m-splitting ultraquasi-pseudometric q on X, $q \leq d$ implies $q = d$, the ultraquasi-pseudometric d is said to be $U(X)$*-minimally m-splitting.*
3. If $\leq$ is an m-produced partial order on X such that there is no partial order $\preceq$ on X having both the properties,
   (a) $\leq \subsetneq \preceq$ (strict inclusion).
   (b) $(X, m, \preceq)$ is produced by an ultraquasi-pseudometric on X,

   then $\leq$ is said *maximally m-produced.*

Remark 2.1.8.

- An ultraquasi-pseudometric d on X which is m-splitting is an ultraquasi-metric (if d is m-splitting then $d^s = m$, a metric).
- For a $U(X)$-minimally m-splitting ultraquasi-pseudometric d (implicitly d is an ultraquasi-metric since it is m-splitting), the conjugate d^{-1} is $U(X)$-minimally m-splitting.
  Indeed, if q is an m-splitting ultraquasi-pseudometric on X with $q \leq d^{-1}$, then for $x, y \in X$, $q(x, y) \leq d^{-1}(x, y)$. That is, for $x, y \in X$, $q^{-1}(y, x) \leq d(y, x)$. Therefore $q^{-1} \leq d$ on X. Therefore, since d is $U(X)$-minimally m-splitting, $q^{-1} = d$. Hence $q = d^{-1}$.
- If $\leq_d^*$ is the *dual order* of $\leq_d$ (implicitly d is considered an ultraquasi-metric here, we traditionally refer to the dual with respect to a partial order), then $\leq_d^* = \leq_{d^{-1}}$.
- If $(X, \leq)$ and $(X, \preceq)$ are partially ordered spaces such that $\leq \subseteq \preceq$, then $\leq^* \subseteq \preceq^*$.

- Given a maximally m-produced partial order over X, its dual order is maximally m-produced too.
  Indeed, suppose that $\leq$ is maximally m-produced by an ultraquasi-pseudometric on X, and suppose that its dual $\leq^*$ is not maximally m-produced. Then there exists $\preceq$ over X such that $\leq^* \subsetneq \preceq$ and $(X, m, \preceq)$ is produced by an ultraquasi-pseudometric on X. Therefore, $\leq \subsetneq \preceq^*$ and $(X, m, \preceq^*)$ is produced by a quasi-pseudometric, contradicting the fact that $\leq$ is maximally m-produced.
- We will see that the notion of maximally m-produced partial orders above is kind of irrelevant when speaking about ultraquasi-metrics in comparison with using the notion with quasi-metrics. We will see that the maximally m-produced partial orders, in terms of ultrametrics, are just the total orders.

Example 2.1.9. For a partially ordered ultra-metric space $(X, m, \leq)$, if d_1, d_2, d_3 are ultraquasi-metrics defined over X such that $d_1 \leq d_2 \leq d_3$ with d_1, d_3 both producing $(X, m, \leq)$, then d_2 also produces $(X, m, \leq)$.

Proof. Since we have $d_1 \leq d_2 \leq d_3$ and $\leq_{d_1} = \leq_{d_3} = \leq$, from what we have pointed out earlier, by passing to specialization orders, we have $\leq_{d_1} \subseteq \leq_{d_2} \subseteq \leq_{d_3}$; and $\leq_{d_2} = \leq$. Also, since $d_1^s = d_3^s = m$, passing to symmetrization gives us $d_2^s = m$. $\square$

2.2 Some Simple Examples

Example 2.2.1. Consider $X = \mathbb{R}$. A rather simple ultrametric over X is defined by $n : X \times X \to \mathbb{R}_+$ such that for $(x, y) \in X \times X$ we have $n(x, y) = |x| \vee |y|$ and $n(x, y) = 0$ for $x = y$.

Now, let $u : X \times X \to \mathbb{R}_+$ such that for $x, y \in X$ we have $u(x, y) = 0$ if $x \leq y$ and $u(x, y) = n(x, y) = |x| \vee |y|$ if $x \not\leq y$ (That is $x > y$). We then need to show that the strong triangle inequality holds, that is we have $u(x, z) \leq u(x, y) \vee u(y, z)$ for $x, y, z \in X$.

Of course we only have to check for the case where $x \not\leq z$ since it is trivially verified for the case of $x \leq z$:

Case 1 for $x \not\leq y$ and $y \not\leq z$, we have $z < y < x$. But then $|x| \vee |z| \leq |x| \vee |y| \vee |z|$, giving us the strong triangle inequality.

Case 2 for $x \not\leq y$ and $y \leq z$, we have $y \leq z < x$. We then have $|z| \leq |x| \vee |y|$, hence $|x| \vee |z| \leq |x| \vee |y|$, which entails the wanted inequality.

Case 3 The last case is when $x \leq y$ and $y \nleq z$. That is $z < x \leq y$. Therefore $|x| \leq |y| \vee |z|$, thus $|x| \vee |z| \leq |y| \vee |z|$.

Hence the strong triangle inequality holds in every single case and therefore u is an ultraquasi-metric over X.

Example 2.2.2. Consider $X = \{a,\, b,\, c\}$, and define the order $\leq$ on X such that $a < b < c$. Then $(X,\, \leq)$ is a totally ordered set. Now, define m to be the ultrametric on X such that $m(a, b) = m(b,\, c) = 2$ and $m(a,\, c) = 1$. We have seen that m cannot be produced by any quasi-metric from [3], hence it cannot be produced by any ultraquasi-metric.

The next results are from [3], but here we are giving out more details. Also notice that even though Gaba and Künzi have made the proofs with quasi-metrics, it is quite easy to switch to ultraquasi-metrics.

Proposition 2.2.3. *Let $(X,\, \leq)$ be a partially ordered set. Then the map,*

$$\begin{aligned} d_{\leq} : X \times X &\longrightarrow \{0,\, 1\} \\ (x,\, y) &\longmapsto d_{\leq}(x,\, y) = \begin{cases} 0, & \text{if } x \leq y, \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

is an ultraquasi-metric on X with $\leq_{(d_{\leq})} = \leq$.

Proof. Let $x,\, y \in X$. If $x \leq y$ then the strong triangle inequality is trivially proved for $d_{\leq}$. If $(x,\, y) \notin \leq$ then $d_{\leq}(x,\, y) = 1$. Let $z \in X$ and suppose we have $(x,\, z),\, (z,\, y) \in \leq$, i.e. $x \leq z$ and $z \leq y$. Then $x \leq y$ by transitivity i.e. $d_{\leq}(x,\, y) = 0$, a contradiction. Necessarily, for $(x,\, y) \notin \leq$, for $z \in X$, $(x,\, z) \notin \leq$ or $(z,\, y) \notin \leq$. Hence the strong triangle inequality is proved in every case.
And by the definition of $d_{\leq}$, we have $\leq_{(d_{\leq})} = \leq$ and $d_{\leq}$ is an ultraquasi-metric. $\square$

Also, note that for an ultraquasi-metric space $(X,\, d)$ such that for $(x,\, y) \in X$, $d(x,\, y) \in \{0,\, 1\}$, we have $d_{(\leq_d)} = d$.
Therefore, if we set

$$A = \{d : X \times X \longrightarrow \{0,\, 1\} \,|\, d \text{ ultraquasi-metric defined on } X\}, \tag{2.5}$$

and

$$B = \{\leq \;|\; \leq \text{ is a partial order on } X\},$$

then the maps $a : d \in A \longmapsto \leq_d \in B$ and $b : \leq \in B \longmapsto d_{\leq} \in A$ are such that

$$b \circ a = \mathrm{Id}_A \text{ and } a \circ b = \mathrm{Id}_B.$$

Hence, the set of all partial orders defined on a set X can be identified with the collection $A = \{d : X\times X \longrightarrow \{0,\, 1\} \,|\, d$ ultraquasi-metric defined on $X\}$.

In the definition of $d_{\leq}$, the value 1 can be replaced with any value in $\mathbb{R}_+^*$. We will see in the next example that this default value of 1 may just help us encounter some already well-known objects, and may help us derive a few observations. In fact, we will even see a version of it with the value 1 replaced with ∞ later.

Example 2.2.4. For $(X,\, \leq)$ a partially ordered space, the ultraquasi-metric $d_{\leq}$ produces $(X,\, d_=,\, \leq)$ with $d_=$ being the trivial ultrametric of X.

Proof. Indeed, for $x,\, y \in X$,

$$d^s_{\leq}(x,\, y) = d_{\leq}(x,\, y) \vee d_{\leq}(y,\, x).$$

We have $d^s_{\leq}(x,\, x) = 0$ by reflexivity of $\leq$. If $x \neq y$, then at least one of $(x,\, y)$ and $(y,\, x)$ does not belong to $\leq$, therefore $d^s_{\leq}(x,\, y) = 1$. $\square$

This example then means that for every partial order $\leq$ on X, the quasi-metrics $d_{\leq}$ is $d_=$-splitting.

Notice that if $\leq_1$ and $\leq_2$ are partial orders on X, then $\leq_1 \subseteq \leq_2$ if and only if $u_{\leq_1} \geq u_{\leq_2}$. Indeed, we have, $\leq_1 \subseteq \leq_2$ if and only if for $(x,\, y) \in X \times X$, $u_{\leq_1}(x,\, y) = 0 \Rightarrow u_{\leq_2}(x,\, y) = 0$.

Proposition 2.2.5. *Let X be a set. If $\leq$ is a total order on X, then $d_{\leq}$ is $U(X)$-minimally $d_=$-splitting.*

Proof. Suppose q an ultraquasi-metric $d_=$-splitting on X, such that $q \leq d_{\leq}$. Since $q \leq d_{\leq} \Leftrightarrow \leq_q \supseteq \leq$, and since $\leq$ is a total order on X, we must have $\leq_q = \leq$. Hence $q = d_{\leq}$. $\square$

Therefore, the Szpilrajn's Extension Theorem which states that every partial order can be extended into a total order might be reformulated as follows.

Theorem 2.2.6. *For a set X, given the collection $A = \{d : X \times X \longrightarrow \{0,\, 1\} \,|\, d$ ultraquasi-metric on $X\}$; for $d \in A$, there exists a $U(X)$-minimally $d_=$-splitting ultraquasi-metric $q \in A$ such that $q \leq d$.*

Note that if we consider the collection A (2.5) above, any $U(X)$-minimally $d_=$-splitting ultraquasi-metric $q \in A$ has a total order as its specialization partial order. This comes from applying Szpilrajn's Extension Theorem original form to $\leq_q$, and then applying the $d_=$-splitting minimality of q.

Therefore the q's in the Theorem 2.2.6 have a total order as their specialization partial order.

2.3 The Bicompletion

As a continuation of the previous set of examples, here, we will just be reviewing the construct of bicompletion as we think that it is good for warming up before we actually jump into our actual subject, and we think that the construction is also nonetheless interesting.

First, let us state a very simple lemma on ultraquasi-pseudometrics which is going to serve us both here and in a latter section.

Lemma 2.3.1. *For an ultraquasi-pseudometric space* (X, q), *we have*

$$|d(x, y) - d(a, b)| \leq d^s(x, a) \vee d^s(y, b) \quad \textit{for } x, y, a, b \in X.$$

Proof. Since the left hand side of the inequality equals to $d(x, y) - d(a, b)$ or $d(a, b) - d(x, y)$, we may just assume $d(a, b) \leq d(x, y)$. So, from the triangle inequality and the strong triangle inequality, we have $d(x, y) \leq d(x, a) + d(a, b) \vee d(b, y)$. So that $d(x, y) - d(a, b) \leq d(x, a) \vee d(b, y)$. □

Definition 2.3.2. An ultraquasi-pseudometric space (X, d) is said *bicomplete* if the ultrapseudometric space (X, d^s) is complete.

Definition 2.3.3. A bicompletion of an ultraquasi-metric space (X, d) is defined as a couple $((\hat{X}, \hat{d}), i)$ such that $(\hat{X}, \hat{d})$ is a bicomplete ultraquasi-metric space and $i : (X, d) \longrightarrow (\hat{X}, \hat{d})$ is an isometry with $i(X)$ being $\tau((\hat{d})^s)$-dense in $\hat{X}$.

Theorem 2.3.4. *Every ultraquasi-metric space* (X, d) *admits a unique bicompletion* $(\hat{X}, \hat{d})$ *up to an isometric isomorphism.*

Proof. Let (X, d) be an ultraquasi-metric space.

We are only going to show the existence of the bicompletion. Let $\mathcal{C}$ be the set of all d^s-Cauchy sequences of points of X and let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be elements of $\mathcal{C}$.

Hence, for $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n, p \geq N$ we have $d^s(x_n, x_p) \leq \epsilon$ and $d^s(y_n, y_p) \leq \epsilon$. Thus, by using Lemma 2.3.1, we get

$$|d(x_n, y_n) - d(x_p, y_p)| \leq d^s(x_n, x_p) \vee d^s(y_n, y_p) \leq \epsilon. \tag{2.6}$$

Therefore the sequence $(d(x_n, y_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ and admits a limit.

So, we can define $\delta : \mathcal{C} \times \mathcal{C} \longrightarrow \mathbb{R}$ such that $\delta((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = \lim_{n\to\infty} d(x_n, y_n)$. We easily see that $(\mathcal{C}, \delta)$ is an ultraquasi-pseudometric space. Hence we can consider the ultra-pseudometric space $(\mathcal{C}, \delta^s)$. Let $\mathcal{R}$

be the equivalence relation defined over $\mathcal{C}$ such that $(x_n)_{n\in\mathbb{N}}\mathcal{R}(y_n)_{n\in\mathbb{N}}$ if and only if $\delta^s((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = 0$.

Thus, let $\hat{X} = {}^{\mathcal{C}}/_{\mathcal{R}}$ and let $\pi : \mathcal{C} \longrightarrow \hat{X}$ be the canonical projection. Then denote $\widehat{(x_n)_{n\in\mathbb{N}}} = \pi((x_n)_{n\in\mathbb{N}})$. Let $(a_n)_{n\in\mathbb{N}} \in \widehat{(x_n)_{n\in\mathbb{N}}}$ and $(b_n)_{n\in\mathbb{N}} \in \widehat{(y_n)_{n\in\mathbb{N}}}$, we want to show that $\delta((a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}) = \delta((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}})$. By assumption, we then have $\lim_{n\to\infty} d(a_n, x_n) = 0 = \lim_{n\to\infty} d(x_n, a_n)$ and $\lim_{n\to\infty} d(b_n, y_n) = 0 = \lim_{n\to\infty} d(y_n, b_n)$. From the strong triangle inequality, we have $d(a_n, b_n) \leq d(a_n, x_n) \vee d(x_n, y_n) \vee d(y_n, b_n)$ and $d(x_n, y_n) \leq d(x_n, a_n) \vee d(a_n, b_n) \vee d(b_n, y_n)$. By passing to limits, we have $\lim_{n\to\infty} d(a_n, b_n) = \lim_{n\to\infty} d(x_n, y_n)$; which shows that $\delta((a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}) = \delta((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}})$.

Therefore, the map $\hat{d} : \hat{X} \times \hat{X} \longrightarrow \mathbb{R}$ given by $\hat{d}(\widehat{(x_n)_{n\in\mathbb{N}}}, \widehat{(y_n)_{n\in\mathbb{N}}}) = \delta((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}})$ is well defined (it does not depend on the elements representing the classes) and it is an ultraquasi-pseudometric by its definition. Moreover, it is an ultraquasi-metric since $(\hat{d})^s$ is an ultrametric on $\hat{X}$ using the definition of our equivalence relation $\mathcal{R}$ above.

Next, define $i : (X, d) \longrightarrow (\hat{X}, \hat{d})$ such that $i(x) = \widehat{(x_n)_{n\in\mathbb{N}}}$ where $(x_n)_{n\in\mathbb{N}}$ is the constant sequence such that $x_n = x$ for all $n \in \mathbb{N}$. Hence, $\hat{d}(i(x), i(y)) = \lim_{n\to\infty} d(x_n, y_n) = d(x, y)$, thus i is an isometry; it is also injective since $\hat{d}$ is an ultraquasi-metric.

Also, $i(X)$ is $\tau((\hat{d})^s)$-dense in $\hat{X}$. Indeed, let $\hat{a} = \widehat{(a_n)_{n\in\mathbb{N}}} \in \hat{X}$ and $\epsilon > 0$. Then $(a_n)_{n\in\mathbb{N}}$ is a d^s-Cauchy sequence so there exists $N \in \mathbb{N}$ such that for $n, p \geq N$, we have $d^s(a_n, a_p) \leq \epsilon$. By fixing $n \geq N$ and making $p \to \infty$, we get $\lim_{p\to\infty} d^s(a_n, a_p) \leq \epsilon$, i.e. $(\hat{d})^s(i(a_n), \hat{a}) = \lim_{p\to\infty} d^s(a_n, a_p) \leq \epsilon$. So $\lim_{n\to\infty} i(a_n) = \hat{a}$ in the ultrametric space $(\hat{X}, (\hat{d})^s)$.

Next, we want to show that the ultrametric space $(\hat{X}, (\hat{d})^s)$ is complete. So, let $(\widehat{a^p})_{p\in\mathbb{N}}$ be a $(\hat{d})^s$-Cauchy sequence of points of $\hat{X}$. Since $i(X)$ is dense in $(\hat{X}, (\hat{d})^s)$, then for $p \in \mathbb{N}^*$ and $\epsilon_p = \frac{1}{p}$, there exists $x_p \in X$, such that $(\hat{d})^s(\widehat{a^p}, i(x_p)) \leq \frac{1}{p}$. In this way, we then obtain a sequence $\alpha = (x_p)_{p\in\mathbb{N}^*}$ of points of X.

Let us show that α is a d^s-Cauchy sequence. Let $\epsilon > 0$. Since the sequence $\left(\frac{1}{p}\right)_{p\in\mathbb{N}^*}$ goes to zero and the sequence $(\widehat{a^p})_{p\in\mathbb{N}}$ is a $(\hat{d})^s$-Cauchy sequence, then there exists $N \in \mathbb{N}$ such that for $k, l \geq N$, we both have $\frac{1}{k} \leq \epsilon$ and $(\hat{d})^s(\widehat{a^k}, \widehat{a^l}) \leq \epsilon$. But by using the isometric property of i and the triangle inequality on $(\hat{d})^s$, we have $d^s(x_k, x_l) = (\hat{d})^s(i(x_k), i(x_l)) \leq (\hat{d})^s(i(x_k), \widehat{a^k}) \vee (\hat{d})^s(\widehat{a^k}, \widehat{a^l}) \vee (\hat{d})^s(\widehat{a^l}, i(x_l)) \leq 3\epsilon$—showing α is a d^s-Cauchy sequence.

Finally, let us show that $\widehat{a^p} \to \hat{\alpha}$ when $p \to \infty$. Let $\epsilon > 0$. Hence there exists $N \in \mathbb{N}$ such that for $k, l \geq N$, we have $\frac{1}{k} \leq \epsilon$ and $d^s(x_k, x_l) \leq \epsilon$. So, by fixing k and by making $l \to \infty$, we get $(\hat{d})^s(i(x_k), \widehat{\alpha}) = \lim_{l\to\infty} d^s(x_k, x_l) \leq \epsilon$. Hence, by the strong triangle inequality and the way how we defined α, we have $(\hat{d})^s(\widehat{a^k}, \hat{\alpha}) \leq (\hat{d})^s(\widehat{a^k}, i(x_k)) \vee (\hat{d})^s(i(x_k), \hat{\alpha}) \leq \frac{1}{k} \vee \epsilon \leq \epsilon$. Thus we showed the existence. □

2.4 A focus on Ultraquasi-metrics

This section here is motivated by similar investigations due to Gaba and Kunzi [3]. Analogously to what they have established, here we are going to focus mainly on ultra-metrics and their splitting. Also, notice that throughout, given a space X, if S is a family of ultraquasi-metrics over X, then the infinimum $\wedge_{d\in S} d$ is taken in the set of all ultraquasi-metrics defined over the space X unless otherwise stated.

Proposition 2.4.1. *Let (X, m) be an ultrametric space and q an m-splitting ultraquasi-metric on X. Then there exists a $U(X)$-minimally m-splitting ultraquasi-metric q_0 on X such that $q_0 \leq q$.*

Proof. Let I be a set of indices and suppose $(d_i)_{i\in I}$ a chain of ultraquasi-pseudometrics defined on X such that, for $i \in I$, $d_i \leq q$ and $d_i^s = m$. Define $d = \inf_{i\in I} d_i$.

We want to show that d is an ultraquasi-pseudometric on X with $d^s = m$. For $i \in I$ and $x \in X$, we have $0 \leq d(x, x) \leq d_i(x, x) = 0$, hence $d(x, x) = 0$. Now, suppose that d does not verify the strong triangle inequality, i.e. suppose that there exists $x, y, z \in X$ such that, $d(x, y) \not\leq d(x, z) \vee d(z, y)$. Hence, there exists $\epsilon > 0$ such that

$$d(x, z) \vee d(z, y) + \epsilon < d(x, y). \tag{2.7}$$

On the other hand, since $d = \inf_{i\in I} d_i$, by definition of the infimum, there exist $i, j \in I$ such that

$$d(x, z) < d_i(x, z) < d(x, z) + \epsilon \quad \text{and} \quad d(z, y) < d_j(z, y) < d(z, y) + \epsilon.$$

But we know that the chain $(d_i)_{i\in I}$ is totally ordered; therefore, $d_i \leq d_j$ or $d_j \leq d_i$. Thus, by setting $k = \min(d_i, d_j)$, we get

$$d_k(x, z) < d(x, z) + \epsilon \quad \text{and} \quad d_k(z, y) < d(z, y) + \epsilon.$$

By taking the supremum, we then get

$$d_k(x, z) \vee d_k(z, y) < d(x, z) \vee d(z, y) + \epsilon$$

And since d_k is an ultraquasi-pseudometric, we get

$$d_k(x,\, y) < d(x,\, z) \vee d(z,\, y) + \epsilon$$

This last inequality with (2.7) gives us $d_k(x,\, y) < d(x,\, y)$, contradicting the fact that $d = \inf_{i \in I} d_i$. Therefore, d is an ultraquasi-pseudometric defined on X with $d \leq q$.

Now, suppose that $m \neq d \vee d^{-1}$. Then necessarily $d \vee d^{-1} < m$. Hence, there exists $(x,\, y) \in X \times X$ such that $d(x,\, y) < m(x,\, y)$ and $d(y,\, x) < m(y,\, x)$. And again, by definition of the infimum, there exist $i,\, j \in I$ such that

$$d(x,\, y) < d_i(x,\, y) < m(x,\, y) \quad \text{ and } \quad d(x,\, y) < d_j(y,\, x) < m(y,\, x)$$

And since $d_i \leq d_j$ or $d_j \leq d_i$, we can define $k = \min(d_i,\, d_j)$. Thus we get, $d_k(x,\, y) < m(x,\, y)$ and $d_k(y,\, x) < m(y,\, x)$; so $d_k^s(x,\, y) < m(x,\, y)$, contradicting the fact that d_k is m-splitting. Hence $d = \inf_{i \in I} d_i$ is ultraquasi-metric defined on X with $d^s = m$ and $d \leq q$.

Therefore the partially ordered set of all ultraquasi-pseudometrics defined over X which are smaller than q and m-splitting, satisfy Zorn's Lemma condition. Thus, the existence of a $U(X)$-minimally m-splitting ultraquasi-metric q_0 on X with $q_0 \leq q$ follows. $\square$

Notice here that given a nonempty totally ordered transfinite sequence of ultraquasi-pseudometrics $(d_i)_{i \in I}$ over a set X, $\inf_{i \in I} d_i$ is also an ultraquasi-pseudometric over X.

Remark 2.4.2. Let $(X,\, \leq)$ be a partially ordered set and q an ultraquasi-metric over X. Let us consider the *extended ultraquasi-metric*, $d_{\leq}^{\infty}$,

$$\begin{aligned} d_{\leq}^{\infty} : X \times X &\longrightarrow \{0,\, \infty\} \\ (x,\, y) &\longmapsto d_{\leq}^{\infty}(x,\, y) = \begin{cases} 0, & \text{if } x \leq y, \\ \infty & \text{otherwise.} \end{cases} \end{aligned}$$

The map $d_{\leq}^{\infty}$ is an ultraquasi-metric on X with $\leq_{(d_{\leq}^{\infty})} = \leq$.

Now, define $r_q = \min(q,\, d_{\leq}^{\infty})$. In general r_q is not an ultraquasi-pseudometric. Obviously, for $x \in X$, $r_q(x,\, x) = 0$. But if we consider the partial ordered set of Example 2.2.2 and choose $q = m$, the ultrametric of the example, then we find that $r_q(b,\, a) = 2$ and $r_q(b,\, c) \vee r_q(c,\, a) = 0$. Hence the strong triangle inequality is not satisfied for r_q.

For $x,\, y \in X$, let $w = w_0 w_1 \cdots w_n$ (with $n \geq 0$) be a word/path made of elements $w_0,\, w_1,\, \ldots,\, w_n \in X$ such that $w_0 = x$ and $w_n = y$. We then define the set

$$N(x,\, y) = \{s_w = \bigvee_{i=0}^{n-1} r_q(w_i,\, w_{i+1}) \,|\, w \text{ a word starting with } x \text{ and ending with } y\}. \tag{2.8}$$

And let $|w| = n+1$, the length of w; and $w[i] = w_i$ for $i \leq n$. And we define

$$U_{(q,\,\leq)}(x,\, y) = \inf N(x,\, y) \tag{2.9}$$

then the map $U_{(q,\,\leq)} : X \times X \longrightarrow \mathbb{R}_+$ is obviously an ultraquasi-pseudometric. Indeed, as we have stated earlier, for $x \in X$, $r_q(x,\, x) = 0$; then we note that $r_q(x,\, x) \in N(x,\, x)$, thus $U_{(q,\,\leq)}(x,\, x) = 0$. On the other hand, for $x,\, y,\, z \in X$ there exists a word w starting with x and ending with y such that $U_{(q,\,\leq)}(x,\, z) \vee U_{(q,\,\leq)}(z,\, y) = s_w$. Hence the strong triangle inequality holds.

Now if we consider an ultraquasi-pseudometric l defined on X such that $l \leq q$ and $l \leq d_{\leq}^{\infty}$, then $l \leq U_{(q,\,\leq)}$. Indeed, we get $l \leq \min(q,\, d_{\leq}^{\infty}) = r_q$. Then using the strong triangle inequality, for $x,\, y,\, z \in X$, we have $l(x,\, y) \leq l(x,\, z) \vee l(z,\, y) \leq r_q(x,\, z) \vee r_q(z,\, y) = s_w$ for some word w starting with x and ending with y. And since $U_{(q,\,\leq)}(x,\, y) = \inf N(x,\, y)$, by definition of the infimum as being the greatest of the lower bounds, we have $l(x,\, y) \leq U_{(q,\,\leq)}(x,\, y)$.

Therefore, any ultraquasi-pseudometric which is smaller than q and $d_{\leq}^{\infty}$, is automatically smaller than the ultraquasi-pseudometric $U_{(q,\,\leq)}$. Hence $U_{(q,\,\leq)} = q \wedge d_{\leq}^{\infty}$, where $\wedge$ corresponds to meet in the partially ordered set of all (extended) ultraquasi-pseudometrics defined on X. Hence $U_{(q,\,\leq)}$ is the greatest of all the ultraquasi-pseudometrics l defined on X, that are lower bounds to q having the property $\leq \subseteq \leq_l$.

Remark 2.4.3. For a partially ordered ultrametric space $(X,\, m,\, \leq)$, the ultraquasi-pseudometric $U_{(m,\,\leq)}$ does not necessarily produce $(X,\, m,\, \leq)$. Again, we can verify this by considering the ordered ultrametric space of Example 2.2.2, which cannot be produced by any ultraquasi-pseudometric.

But in the case that $(X,\, m,\, \leq)$ can be produced by an ultraquasi-metric, then $U_{(m,\,\leq)}$ is the greatest of all ultraquasi-pseudometrics producing it. Indeed, suppose q a quasi-metric producing $(X,\, m\, \leq)$. Then we have $\leq_q = \leq$, and hence $q(x,\, y) = 0$ if and only if $d_{\leq}^{\infty}(x,\, y) = 0$ for $x,\, y \in X$; which then allows us to write $q \leq d_{\leq}^{\infty}$. On the other hand, since q is m-splitting, we are allowed to write $q \leq m$. Therefore, by the point of Remark 2.4.2, we have $q \leq U_{(m,\,\leq)} \leq m$. By passing to symmetrization, this leads to $U_{(m,\,\leq)}^s = m$. And since we already know (again from Remark 2.4.2) that $\leq \subseteq \leq_{U_{(m,\,\leq)}}$, by passing to specialization preorder, with the former last inequality, the specialization orders for q and $U_{(m,\leq)}$ are both equal to $\leq$.

Corollary 2.4.4. *A partially ordered ultrametric space* $(X, m, \leq)$ *is produced by an ultraquasi-metric if and only if* $m \leq U^s_{(m,\leq)}$ *and* $\leq_{U_{(m,\leq)}} \subseteq \leq$.

Proof. Clear by looking at the last 3 steps/sentences of Remark 2.4.3. □

Remark 2.4.5.

- If $\leq$ is a total order on X, then the second part of the condition in Corollary 2.4.4 is not necessary.
- If X is finite, then the second part of the condition is also not necessary. Indeed, because of the finiteness of X, for $x, y \in X$, $U_{(m,\leq)}(x, y) \in N(x, y)$. Therefore, if $U_{(m,\leq)}(x, y) = 0$, by definition of $N(x, y)$ (2.8), there exists a word $w = x_0 \dots x_n$ of elements of X such that,

$$\bigvee_{i=0}^{n-1} \min(m, d^{\infty}_{\leq})(x_i, x_{i+1}) = 0. \tag{2.10}$$

  Since all terms involved in the supremum expression are ≥ 0, necessarily for $i \in I = \{0, \dots, n-1\}$, $\min(m, d^{\infty}_{\leq})(x_i, x_{i+1}) = 0$. Hence for $i \in I$, $m(x_i, x_{i+1}) = 0$ or $d^{\infty}_{\leq}(x_i, x_{i+1}) = 0$. But if there exists $i_0 \in I$ such that $d^{\infty}_{\leq}(x_{i_0}, x_{i_0+1}) \neq 0$, then (2.10) is not finite and therefore is not equal to 0. Thus for $i \in I$, $d^{\infty}_{\leq}(x_i, x_{i+1}) = 0$, i.e. for $i \in I$, $x_i \leq x_{i+1}$. Hence by transitivity of $\leq$, we have $x = x_0 \leq x_n = y$.

Definition 2.4.6. Given a set X, a *dissimilarity* on X is a function $d : X \times X \longrightarrow \mathbb{R}_+$ such that, for $x, y \in X$,

- $d(x, y) = d(y, x)$,
- $d(x, y) = 0$ if and only if $x = y$.

Notice that no triangle inequality is required for a dissimilarity. Hence, given a set X and a dissimilarity defined over it, if the dissimilarity satisfies the strong triangle inequality then it is an ultrametric.

In topology, a dissimilarity is called semi-metric.

Definition 2.4.7. Given a set X and a dissimilarity d on X, if there exists a total ordering $\leq$ over X satisfying the condition,

for $x, y, z \in X$ such that $x \leq y \leq z$, we have,

$$\max(d(x, y), d(y, z)) \leq d(x, z),$$

then the dissimilarity d is called *Robinsonian*.

Therefore, because of the strong triangle inequality, for a Robinsonian ultraquasi-metric, the latter inequality becomes an equality.

Lemma 2.4.8 (Interval condition)**.** *If a totally ordered ultrametric space $(X,\, m,\, \leq)$ can be produced by an ultraquasi-metric, then $\leq$ makes m Robinsonian.*

Proof. Suppose there exists an ultraquasi-metric q producing $(X,\, m,\, \leq)$, and let $x,\, y,\, z \in X$ such that $x \leq y \leq z$. Since $x \leq y$, we have $q(x,\, y) = 0$ and $q(y,\, x) = q^s(x,\, y) = m(x,\, y)$. So, $y \geq x$ implies $q(y,\, x) = m(x,\, y)$. Hence, by analogy, $q(z,\, x) = m(x,\, z)$ and $q(z,\, y) = m(y,\, z)$.

Hence $m(x,\, y) = q(y,\, x) \leq q(y,\, z) \vee q(z,\, x) = 0 \vee m(x,\, z)$; and $m(y,\, z) = q(z,\, y) \leq q(z,\, x) \vee q(x,\, y) = m(x,\, z) \vee 0$. □

Remark 2.4.9. As for a quasi-metric, for an ultraquasi-metric q with partial order $\leq$, we have

$$r_q(x,\, y) = \begin{cases} 0, & \text{if } (x,\, y) \in \leq, \\ q(x,\, y) & \text{otherwise.} \end{cases}$$

Remark 2.4.10. For a partially ordered space $(X,\, m,\, \leq)$ such that $\leq$ is a total order, if $(X,\, m,\, \leq)$ is produced by an ultraquasi-metric q, then $q = r_m = \min(m,\, d^{\infty}_{\leq})$ (compare with [3]).

Proof. By the last bullet of Remark 2.1.8, the ultraquasi-metric q is defined in a unique way such that for $(x,\, y) \in X \times X$, if $(x,\, y) \in \leq$, then $q(x,\, y) = 0$; otherwise $q(x,\, y) = m(x,\, y)$. But that is exactly the definition of r_m (Remark 2.4.9). □

The next results then help us characterize the totally ordered ultrametric spaces that can be produced by an ultraquasi-metric.

Proposition 2.4.11. *Consider an ultrametric space $(X,\, m)$ and let $\leq$ be a total order defined over X. There exists an ultraquasi-metric producing $(X,\, m,\, \leq)$ if and only if $\leq$ makes m Robinsonian (compare with [3]).*

Proof. ($\Rightarrow$) Follows from Lemma 2.4.8, the interval condition.

($\Leftarrow$) Suppose that $\leq$ makes m Robinsonian. From what we have said, we want to show that the function d defined over $X \times X$ such that $d := r_m = \min(m,\, d^{\infty}_{\leq})$, is an ultraquasi-metric and produces $(X,\, m,\, \leq)$.

We easily find that $d^s = m$ and that for $x,\, y \in X$, we have $x \leq y$ if and only if $d(x,\, y) = 0$. What remains is then to verify the strong triangle inequality.

Let $x,\, y,\, z \in X$. We want to show that $d(x,\, z) \leq d(x,\, y) \vee d(y,\, z)$. If $x \leq z$, then $d(x,\, z) = 0$ and the strong triangle inequality is trivially verified. If $z \leq x$, then we have 3 cases:

- If $z \le y \le x$, then the strong triangle inequality applied to m gives us $d(x, z) = m(x, z) \le m(x, y) \vee m(y, z) \le d(x, y) \vee d(y, z)$.
- If $y \le z \le x$, since $\le$ makes m Robinsonian, then $d(x, z) = m(z, x) \le m(y, x) = d(x, y) \vee 0 = d(x, y) \vee d(y, z)$.
- If $z \le x \le y$, again, by the "Robinsonianity" of m with respect to $\le$, we have $d(x, z) = m(z, x) \le m(z, y) = 0 \vee d(y, z) = d(x, y) \vee d(y, z)$. $\square$

Remark 2.4.12. Considering a partially ordered space $(X, m, \le)$ that can be produced by an ultraquasi-metric, we are going to give here another proof for Lemma 2.4.8, the interval condition. We are going to use the condition $m \le U^s_{(m,\le)}$ of Corollary 2.4.4 (So $U^s_{(m,\le)} = m$).

Proof. Define $D := U_{(m,\le)}$ and consider $r_m = \min(m, d^\infty_{\le})$. Let $x, y, z \in X$ such that $x \le y \le z$. Since by assumption $m \le D^s$,

$$\begin{aligned} m(x, y) &\le \max(D(x, y), D(y, x)),\\ &\le \max(r_m(x, x) \vee r_m(x, y), r_m(y, z) \vee r_m(z, x)) \text{ (by (2.8) and (2.9))},\\ &\le r_m(z, x) \text{ (because } r_m(x, y) = r_m(y, z) = r_m(x, x) = 0), \end{aligned}$$

Since $z \not\le x$, we have $r_m(z, x) = m(z, x)$. Therefore $m(x, y) \le m(z, x)$. In a similar fashion, $m(y, z) \le \max(D(y, z), D(z, y)) \le \max(r_m(y, z) \vee r_m(z, z), r_m(z, x) \vee r_m(x, y)) \le r_m(z, x) = m(z, x)$. $\square$

Remark 2.4.13. Before we state the next result, let us first point out a few things:

- First, remember that for any partial orders $\le$ and $\preceq$, defined on a set X; $\le \subseteq \preceq$ if and only if $d^\infty_{\le} \ge d^\infty_{\preceq}$ (proof same as earlier).

  So, given an ultraquasi-metric q over X, we have $U_{(q,\le)} \ge U_{(q,\preceq)}$.
- Next, consider over a set X a totally ordered set of partial orders $(\le_i)_{i\in I} \subseteq X \times X$. Notice that $\bigcup_{i\in I} \le_i$ is a partial order on X, and also that $\bigcup_{i\in I} \le_i = \bigvee_{i\in I} \le_i$ for the inclusion. Therefore, from the first bullet, for any $j \in I$ we have $d^\infty_{\le_j} \ge d^\infty_{\bigvee_{i\in I}\le_i}$.

  Now, let q be an ultraquasi-pseudometric defined over X such that $q \le d^\infty_{\le_j}$ for all $j \in I$. Hence, for any $j \in I$ and $(x, y) \in \le_j$, we have $q(x, y) = 0$. Thus for $(x, y) \in \bigvee_{i\in I} \le_i$, $q(x, y) = 0$. Therefore $q \le d^\infty_{\bigvee_{i\in I}\le_i}$ by the definition of $d^\infty_{\bigvee_{i\in I}\le_i}$.

  So, $\bigwedge_{i\in I} d^\infty_{\le_i} = d^\infty_{\bigvee_{i\in I}\le_i}$.
- Given an arbitrary set of ultraquasi-pseudometrics $(d_i)_{i\in I}$ defined over a set X, the function $f := \inf_{i\in I} d_i$, is not necessarily an ultraquasi-pseudometric (we have seen that when we defined r_q for an ultraquasi-metric q, Remark 2.4.2). Also, notice that since $g := \bigwedge_{i\in I} d_i$ always

exists (the meet is taken in the lattice of all ultraquasi-pseudometrics), then $g \leq f$ (since all of the d_i's are greater than g, thus their infimum too). Hence if the function f is an ultraquasi-pseudometric, by the definition of g and since f is smaller than all of the d_i's, we have $f \leq g$, therefore $f = g$, i.e. $\inf_{i\in I} d_i = \bigwedge_{i\in I} d_i$ (Compare with Proposition 2.4.1).

Proposition 2.4.14. *For an ultrametric space (X, m), every maximally m-produced partial order over X is the specialization order of a $U(X)$-minimally m-splitting ultraquasi-metric on X.*

Proof. Let $\leq$ be a maximally m-produced partial order over X and let q be an ultraquasi-metric producing $(X, m, \leq)$. By Proposition 2.4.1, there exists a $U(X)$-minimally m-splitting ultraquasi-metric q_0 over X such that $q_0 \leq q$. Hence, $\leq_q = \leq \subseteq \leq_{q_0}$. And since $\leq$ is maximally m-produced, $\leq_{q_0} = \leq$. □

Example 2.4.15. Consider again the totally ordered ultra-metric space from Example 2.2.2.

- We have seen that the totally ordered ultra-metric space could not be produced by any ultraquasi-metric. This can also be explained by the fact that the totally ordered ultra-metric space is not Robinsonian.
- However, if we consider the modified space $(X, m \wedge 1, \leq)$, this one is produced by $d_{\leq} \wedge 1$.

2.5 An Interesting Ultraquasi-metric

What we are going to see next is a particular ultraquasi-metric which enables us "to split any ultraquasi-metric" in some sense. We are going to elaborate on the method used in [3] and modified to our setting.

Lemma 2.5.1. *Let (X, d) be an ultraquasi-metric space and $a, b \in X$ be elements that are not comparable with $\leq_d$ (that is $d(a, b) > 0$ and $d(b, a) > 0$). Let $l \in [0, d(a, b))$. Let $u_{<ab,l>} : X \times X \longrightarrow \mathbb{R}_+$ be a function such that*

$$u_{<ab,l>}(x, y) = \min(d(x, a) \vee l \vee d(b, y), d(x, y)), \quad x, y \in X.$$

(If $l = 0$, we just write $u_{<ab>}$ instead of $u_{<ab,0>}$)

Then $u_{<ab,l>}$ defines an ultraquasi-metric over X such that $u_{<ab,l>} < d$, also $u_{<ab>} \vee u_{<ba>} = d$, and finally $u_{<ab,l>}$ is the greatest of all ultraquasi-metrics q over X having the properties $q(a, b) = l$ and $q \leq d$.

Proof. Let $x, y, z \in X$. Obviously, $u_{<ab,l>}(x, x) = 0$. For the triangle inequality, we want to show

$$\begin{aligned}\min(d(x, a) \vee l \vee d(b, y), d(x, y)) \leq &\min(d(x, a) \vee l \vee d(b, z), d(x, z)) \\ &\vee \min(d(z, a) \vee l \vee d(b, y), d(z, y))\end{aligned}$$

Let us denote by $k(x, y, z)$ the right-hand side of the inequality. Then four cases can happen.

Case 1 : $k(x, y, z) = d(x, z) \vee d(z, y) \geq d(x, y) \geq u_{<ab,l>}(x, y)$.

Case 2 : $k(x, y, z) = d(x, a) \vee l \vee d(b, z) \vee d(z, y) \geq d(x, a) \vee l \vee d(b, y) \geq u_{<ab,l>}(x, y)$.

Case 3 : $k(x, y, z) = d(x, z) \vee d(z, a) \vee l \vee d(b, y) \geq d(x, a) \vee l \vee d(b, y) \geq u_{<ab,l>}(x, y)$.

Case 4 : $k(x, y, z) = d(x, a) \vee l \vee d(b, z) \vee d(z, a) \vee l \vee d(b, y) \geq d(x, a) \vee l \vee d(b, y) \geq u_{<ab,l>}(x, y)$.

Therefore, $u_{<ab,l>}$ is an ultraquasi-pseudometric over X. To show that $u_{<ab,l>}$ is an ultraquasi-metric, we just need to show that $u_{<ab>}$ is an ultraquasi-metric (since $u_{<ab>} \leq u_{<ab,l>}$). Hence, suppose that $u_{<ab>}(x, y) = 0 = u_{<ab>}(y, x)$. This is equivalent to $u_{<ab>}(x, y) = \min(d(x, a) \vee d(b, y), d(x, y)) = 0$ and $u_{<ab>}(y, x) = \min(d(y, a) \vee d(b, x), d(y, x)) = 0$. And again, we have four cases,

Case 1 : $u_{<ab>}(x, y) = d(x, y) = 0$ and $u_{<ab>}(y, x) = d(y, x) = 0$. Therefore $x = y$, since d is an ultraquasi-metric.

Case 2 : $u_{<ab>}(x, y) = d(x, y) = 0$ and $u_{<ab>}(y, x) = d(y, a) \vee d(b, x) = 0$. Hence $d(y, a) \vee d(b, x) \vee d(x, y) = 0$. Thus $d(y, a) \vee d(b, y) = 0$ (by the strong triangle inequality). So, $d(b, a) = 0$, contradicting the fact that $(b, a) \notin \leq_d$.

Case 3 : $u_{<ab>}(x, y) = d(x, a) \vee d(b, y) = 0$ and $u_{<ab>}(y, x) = d(y, x) = 0$. Like the preceding case we get $d(b, a) = 0$, a contradiction.

Case 4 : $u_{<ab>}(x, y) = d(x, a) \vee d(b, y) = 0$ and $u_{<ab>}(y, x) = d(y, a) \vee d(b, x) = 0$. Then $d(x, a) \vee d(b, y) \vee d(y, a) \vee d(b, x) = 0$. And again $d(b, a) = 0$, a contradiction.

That is, the only possible case is the first case and that $u_{<ab>}$ is then an ultraquasi-metric. Therefore, $u_{<ab,l>}$ is also an ultraquasi-metric.

Next, we have $u_{<ab,l>}(a, b) = l \in [0, d(a, b))$ and $u_{<ab,l>} \leq d$ (as it is the minimum). Hence $u_{<ab,l>} < d$.

Also, since $d(a, b) > 0$ then we have the ultraquasi-metric $u_{<ba>}$. Then, since $u_{<ab>} \leq d$ and $u_{<ba>} \leq d$, we have $u_{<ab>} \vee u_{<ba>} \leq d$. Suppose that we have $(u_{<ab>} \vee u_{<ba>})(x, y) < d(x, y)$, i.e. $u_{<ab>}(x, y) < d(x, y)$ and $u_{<ba>}(x, y) < d(x, y)$. So, $u_{<ab>}(x, y) = d(x, a) \vee d(b, y) < d(x, y)$ and $u_{<ba>}(x, y) = d(x, b) \vee d(a, y) < d(x, y)$. By taking the supremum/maximum,

$$d(x, a) \vee d(a, y) \vee d(x, b) \vee d(b, y) < d(x, y).$$

Hence by the strong triangle inequality, this will lead to $1 < 1$, nonsense. We then conclude that $u_{<ab>} \vee u_{<ba>} = d$.

We already pointed out above that $u_{<ab, l>}(a, b) = l$. What is left to show is that it is the greatest of all ultraquasi-metrics smaller than d satisfying that property. Let q be an ultraquasi-metric such that $q \leq d$ and $q(a, b) = l$. Then, $q(x, y) \leq q(x, a) \vee q(a, b) \vee q(b, y) \leq q(x, a) \vee l \vee q(b, y)$. Therefore, $q(x, y) \leq d(x, y)$ and $q(x, y) \leq d(x, a) \vee l \vee d(b, y)$. Hence $q \leq u_{<ab, l>}$. We then conclude that $u_{<ab, l>}$ is the supremum (or maximum) of all ultraquasi-metrics q smaller than d satisfying $q(a, b) = l$. □

Remark 2.5.2. The definition of $u_{<ab, l>}$ above, can be extended by allowing $l \in \mathbb{R}_+$ and to not only apply to elements a, b that cannot be compared with $\leq_d$.

Indeed, if $a, b \in X$ are such that $d(a, b) = d(b, a) = 0$ and we allow $l \in \mathbb{R}_+$, then $a = b$ and by the strong triangle inequality we have $u_{<ab, l>} = u_{<aa, l>} = d$.

If $a, b \in X$ are such that $d(a, b) \geq 0$, then if we allow l to be such that $l \geq d(a, b)$, then by the strong triangle inequality we just obtain $u_{<ab, l>} = d$.

We are now going to point out a few observations related to what we have seen in the previous sections.

Remark 2.5.3. Let (X, d) be an ultraquasi-metric space, $a, b \in X$ be elements not comparable with $\leq_d$ (that is $d(a, b) > 0$ and $d(b, a) > 0$).

1. We have

$$u_{<ab>} = U_{(d, \Delta \cup \{(a, b)\})}.$$

Indeed, since we have $U_{(d, \Delta\cup\{(a, b)\})}(x, y) = d \wedge d^{\infty}_{\leq_{\Delta\cup\{(a, b)\}}}$, we have $U_{(d, \Delta\cup\{(a, b)\})}(a, b) = 0$ and $U_{(d, \Delta\cup\{(a, b)\})} \leq d$. Therefore, $U_{(d, \Delta\cup\{(a, b)\})} \leq u_{<ab>}$ by the former Lemma 2.5.1.

On the other hand, Let $x, y \in X$ and let us consider $N(x, y)$ (2.8) with respect to d and $\Delta \cup \{(a, b)\}$, i.e. consider $r_d = \min(d, d^{\infty}_{\Delta\cup\{(a, b)\}})$.

But we have

$$r_d(x,\, y) = \begin{cases} 0, & \text{if } (x,\, y) = (a,\, b) \\ d(x,\, y) & \text{otherwise.} \end{cases}$$

Let $w = w_0 w_1 \cdots w_n$ (with $n \geq 0$) be a word/path made of elements $w_0,\, w_1,\, \ldots,\, w_n \in X$ such that $w_0 = x$ and $w_n = y$. If w does not contain the sub-word ab then

$$s_w = \bigvee_{i=0}^{n-1} d(w_i,\, w_{i+1}) \geq d(x,\, y) \geq u_{<ab>}(x,\, y).$$

If w does contain $p > 0$ sub-words ab ($2p \leq n$), then there exists $p+1$ words (taking into account words of length 0) such that w can be written as

$$w = l_0\, ab\, l_1\, ab\, l_2\, ab \ldots ab\, l_p,$$

with the fact that for $i = 0,\, \ldots,\, p$, the word l_i does not contain the sub-word ab and $|l_i| \geq 0$. Again, we have four cases.

Case 1 : If $|l_0| > 0$ and $|l_p| > 0$, then by the strong triangle inequality

$$d(x,\, a) \leq \bigvee_{i=0}^{|l_0|-2} d(l_0[i],\, l_0[i+1]) \vee d(l_0[|l_0|-1],\, a),$$
$$d(x,\, a) \leq \bigvee_{i=0}^{|l_0|-2} r_d(l_0[i],\, l_0[i+1]) \vee r_d(l_0[|l_0|-1],\, a) \quad \text{and,}$$
$$d(b,\, y) \leq d(b,\, l_p[0]) \vee \bigvee_{i=0}^{|l_p|-2} d(l_p[i],\, l_p[i+1]),$$
$$d(b,\, y) \leq r_d(b,\, l_p[0]) \vee \bigvee_{i=0}^{|l_p|-2} r_d(l_p[i],\, l_p[i+1]).$$

By taking the supremum/maximum of these inequality, we have $d(x,\, a) \vee d(b,\, y) \leq s_w$, which gives $u_{<ab>}(x,\, y) \leq s_w$.

Case 2 : If $|l_0| = 0$ and $|l_p| > 0$, then $x = a$ and $d(x,\, a) \vee d(b,\, y) = d(b,\, y) \leq s_w$, which once again gives $u_{<ab>}(x,\, y) \leq s_w$.

Case 3 : If $|l_0| > 0$ and $|l_p| = 0$, then $y = b$ and $d(x,\, a) \vee d(b,\, y) = d(x,\, a) \leq s_w$, giving us once more $u_{<ab>}(x,\, y) \leq s_w$.

Case 4 : If $|l_0| = |l_p| = 0$, then $x = a$ and $y = b$ and $d(x,\, a) \vee d(b,\, y) = d(a,\, a) \vee d(b,\, b) = 0$; showing us once more that $u_{<ab>}(x,\, y) \leq s_w$.

Therefore, $u_{<ab>}(x, y) \le U_{(d, \Delta\cup\{(a, b)\})}(x, y)$ and the equality holds.

2. We have

   $\le_{(u_{<ab>})} = \le_d \cup \{(x, y) \in X \times X \mid x \le_d a \text{ and } b \le_d y\}$. (Note it strictly contains $\le_d$).

   Indeed, for $x, y \in X$, the expression $u_{<ab>}(x, y) = \min(d(x, a) \vee d(b, y), d(x, y))$ equals to zero if and only if one of the arguments of the function min is zero if and only if $d(x, a) \vee d(b, y) = 0$ or $d(x, y) = 0$ if and only if $d(x, a) = 0$ and $d(b, y) = 0$, or $d(x, y) = 0$ if and only if $x \le_d a$ and $b \le_d y$, or $x \le_d y$.

Example 2.5.4. Let $(X, m, \le)$ be a partially ordered metric space produced by an ultraquasi-metric d and let $a, b \in X$ be elements such that $0 < d(a, b) < d(b, a)$. Then $u_{<ab>}$ is not m-splitting.
Indeed, $u_{<ba>}(b, a) = 0$ and $u_{<ba>}(a, b) = \min(d(a, b) \vee d(a, b), d(a, b)) = d(a, b)$. Yet $m(a, b) = d^s(a, b) = d(b, a)$ and we know that $u_{<ba>} \le d$. Therefore $u^s_{<ba>} < m$.

Example 2.5.5. For an ultrametric space (X, m) and elements $a, b \in X$ such that $a \ne b$, the ultraquasi-metric $m_{<ab>} = U_{(m, \Delta\cup\{(a, b)\})}$ produces the partially ordered ultrametric space $(X, m, \Delta \cup \{(a, b)\})$.

Proof. Since m is an ultra-metric and $a \ne b$, we have $m(b, a) = m(a, b) > 0$ and the ultraquasi-metric $m_{<ab>}$ is defined. Again, by the "metricity" of m, for $x, y \in X$, we have $m_{<ab>}(x, y) = (m_{<ba>})^{-1}(x, y)$; hence $m_{<ab>} = (m_{<ba>})^{-1}$. Therefore, by Lemma 2.5.1, $m = m_{<ab>} \vee m_{<ba>} = m_{<ab>} \vee (m_{<ab>})^{-1} = (m_{<ab>})^s$. Thus $m_{<ab>}$ is m-splitting and the specializaton order of $m_{<ab>}$ is of course $\le_{m_{<ab>}} = \le_{U_{(m, \Delta\cup\{(a, b)\})}} = \Delta \cup \{(a, b)\}$. (Compare with [3]). ☐

Corollary 2.5.6. *For an ultrametric space (X, m), there exists a collection $\mathcal{D}$ of $U(X)$-minimally m-splitting ultraquasi-metrics over X such that we can write $m = \bigvee \mathcal{D}$.*

Proof. By Example 2.5.5, for every $a, b \in X$ such that $a \ne b$, the quasi-metric $m_{<ab>}$ produces $(X, m, \Delta \cup \{(a, b)\})$. Then using Proposition 2.4.1, there exists a $U(X)$-minimally m-splitting quasi-metric $q_{a, b}$ (i.e. an ultraquasi-metric which depends on a and b) on X such that $q_{a, b} \le m_{<ab>}$. Thus $q_{a, b}(a, b) \le m_{<ab>}(a, b) = 0$, i.e. $q_{a, b}(a, b) = 0$. And since $q_{a, b}$ is m-splitting, $m(a, b) = q_{a, b}(b, a)$. Finally, we have

$$m = \bigvee_{\substack{a, b \in X \\ a \ne b}} q_{a, b}.$$

(Compare with [3]). ☐

Proposition 2.5.7. *For a partially ordered ultrametric space* $(X, m, \leq)$ *that can be produced, there exists a total order* $\leq_0$ *on* X *such that* $\leq \subseteq \leq_0$ *and that* $(X, m, \leq_0)$ *can be produced.*

Proof. Let us consider the collection of subsets of $X \times X$,

$$\mathcal{M} = \{\preceq \text{ partial order on } X \mid (X, m, \preceq) \text{ can be produced by an ultraquasi-metric and } \leq \subseteq \preceq\}.$$

Of course, $\mathcal{M}$ is ordered by the inclusion. What we want to show is that a maximal element exists in $\mathcal{M}$. Let I be a set of indices and $(\preceq_i)_{i\in I} \subseteq \mathcal{M}$ be a nonempty totally ordered set in $\mathcal{M}$. Since our set is totally ordered, the subset of $X \times X$ defined by $\bigcup_{i\in I} \preceq_i$ defines a partial order on X.

For every triple $(X, m, \preceq_i)$, consider then its greatest producing quasi-metric; namely $U_{(m, \preceq_i)}$ (Remark 2.4.3). Since for any partial order $\preceq$ on X, we have $U_{(m, \preceq)} = m \wedge d^{\infty}_{\preceq}$, using the first bullet of Remark 2.4.13 gives us: for $i, j \in I$, if $\preceq_i \subseteq \preceq_j$, then $U_{(m, \preceq_i)} \geq U_{(m, \preceq_j)}$. Hence the set of ultraquasi-metrics $(U_{(m, \preceq_i)})_{i\in I}$ is a nonempty totally ordered set of m-splitting ultraquasi-metrics defined over X.

By the second bullet of Remark 2.4.13, the ultraquasi-pseudometric $\bigwedge_{i\in I} U_{(m, \preceq_i)}$ is defined. Moreover, by the fourth bullet of Remark 2.4.13 and by the proof of Proposition 2.4.1, $\bigwedge_{i\in I} U_{(m, \preceq_i)} = \inf_{i\in I} U_{(m, \preceq_i)}$ is an m-splitting ultraquasi-metric producing the space $(X, m, \leq_{\bigwedge_{i\in I} U_{(m, \preceq_i)}})$.

Notice that we have $\preceq_j \subseteq \leq_{\bigwedge_{i\in I} U_{(m, \preceq_i)}}$ for every $j \in I$. Indeed, for $x, y \in X$ and $j \in I$, such that $x \preceq_j y$, we have $U_{(m, \preceq_j)}(x, y) = 0$, and hence $(\bigwedge_{i\in I} U_{(m, \preceq_i)})(x, y) = 0$. Hence $x \leq_{\bigwedge_{i\in I} U_{(m \leq_i)}} y$.

Thus $(\bigcup_{i\in I} \preceq_i) \subseteq \leq_{\bigwedge_{i\in I} U_{(m \preceq_i)}}$ and we have found an element of $\mathcal{M}$, namely $\leq_{\bigwedge_{i\in I} U_{(m \preceq_i)}}$, which is an upper bound to our total ordered set $(\preceq_i)_{i\in I} \subseteq \mathcal{M}$. Therefore, $\mathcal{M}$ satisfies Zorn's Lemma and $\mathcal{M}$ admits a maximal element.

(Compare with [3]). ∎

Proposition 2.5.8. *For any ultrametric space* (X, m)*, we can write*

$$m = \bigvee \{U_{(m, \leq_{ab})} \mid a, b \in X, a \neq b\},$$

with $\leq_{ab}$ *being a total order over* X *containing* (a, b)*. Moreover* $(X, m, \leq_{ab})$ *can be produced.*

Proof. Indeed, by Example 2.5.5, for every $a, b \in X$ such that $a \neq b$, the ultraquasi-metric $m_{<ab>}$ produces $(X, m, \Delta \cup \{(a, b)\})$. And by Proposition 2.5.7, there exists a total order $\leq_{ab}$ such that $\Delta \cup \{(a, b)\} \subseteq \leq_{ab}$ and that $(X, m, \leq_{ab})$ can be produced. Hence, by Remark 2.4.3, the ultraquasi-metric $U_{(m, \leq_{ab})}$ produces $(X, m, \leq_{ab})$ and thus the result.

(Compare with [3]). ∎

Lemma 2.5.9. *Let X be a set and q be an ultraquasi-metric defined over X. Then if $d := U_{(q,\leq_q)}$, we have $\leq_q = \leq_d$.*

Proof. We already know that $\leq_q \subseteq \leq_d$. On the other hand, for $(x, y) \in \leq_d$ we have $d(x, y) = 0$. Suppose then that $(x, y) \notin \leq_q$. Then $q(x, y) \neq 0$ and $d^{\infty}_{\leq_q}(x, y) = \infty$. Of course, for any ultraquasi-metric q_1 on X, we have $q_1 \leq q$ and $q_1 \leq d^{\infty}_{\leq_q}$ if and only if $q_1 \leq q$. Therefore, from the proof of Lemma 2.5.1, we can conclude that for $l \in \mathbb{R}^*_+$ such that $l < q(x, y)$, we have $u_{<xy,l>}(x, y) = l$ and that $d < u_{<xy,l>} < q$—contradicting the fact that d is supposed to be the greatest of all ultraquasi-metrics with specialization order containing that of q. Hence $\leq_d \subseteq \leq_q$. □

Corollary 2.5.10. *Let X be a finite set and q an ultraquasi-metric defined over X. Then there exists a finite collection L of total orders over X, such that $\bigcap L = \leq_q$, every $U_{(q,\leq)}$ ($\leq \in L$) is an ultraquasi-metric, and we can write $q = \bigvee_{\leq \in L} U_{(q,\leq)}$.*

Proof. First, notice that $U_{(q,\leq_q)} = q$ and the collection $L_0 = \{\leq_q\}$ satisfies $\bigcap L_0 = \leq_q$ and $\bigvee_{\leq \in L_0} U_{(q,\leq)} = q$. If $\leq_q$ is a total order we are done.

Suppose that $\leq_q$ is not a total order and denote $d := U_{(q,\leq_q)}$. From Lemma 2.5.9, we have $d = U_{(q,\leq_d)}$.

Since $\leq_d$ is not a total order, there exist $a, b \in X$ not comparable with $\leq_d$, and we have $d = d_{<ab>} \vee d_{<ba>}$ by Lemma 2.5.1. Since $d_{<ab>} \leq d$, we have $\leq_{d_{<ab>}} \supseteq \leq_d$; hence $U_{(q,\leq_{d_{<ab>}})} \leq U_{(q,\leq_d)} = d$ by the first bullet of Remark 2.4.13. On the other hand, from Lemma 2.5.9, again, we have $d_{<ab>} = U_{(d_{<ab>},\leq_{d_{<ab>}})}$. And since $d_{<ab>} \leq d$ we get $d_{<ab>} = U_{(d_{<ab>},\leq_{d_{<ab>}})} \leq U_{(q,\leq_{d_{<ab>}})}$. Therefore we have the inequality

$$d_{<ab>} \leq U_{(q,\leq_{d_{<ab>}})} \leq d. \tag{2.11}$$

Similarly, we have $d_{<ba>} \leq U_{(q,\leq_{d_{<ba>}})} \leq d$. With these inequality we can get the supremum so that

$$d = d_{<ab>} \vee d_{<ba>} \leq U_{(q,\leq_{d_{<ab>}})} \vee U_{(q,\leq_{d_{<ab>}})} \leq d.$$

Hence $U_{(q,\leq_{d_{<ab>}})} \vee U_{(q,\leq_{d_{<ab>}})} = d$. Moreover, with that last expression, we see that $\leq_d = \leq_{U_{(q,\leq_{d_{<ab>}})}} \cap \leq_{U_{(q,\leq_{d_{<ba>}})}}$.

From inequality (2.11), $U_{(q,\leq_{d_{<ab>}})}$ is an ultraquasi-metric. Similarly, $U_{(q,\leq_{d_{<ba>}})}$ is an ultraquasi-metric.

Moreover, from inequality (2.11) and the fact that $U_{(q,\leq_{d_{<ab>}})} = q \wedge d^{\infty}_{\leq_{d_{<ab>}}}$, we deduce that $U_{(q,\leq_{d_{<ab>}})}$ and $d_{<ab>}$ have obviously the same specialization orders. Similarly, $U_{(q,\leq_{d_{<ba>}})}$ and $d_{<ba>}$ have the same specialization orders.

Hence the collection $L_1 = \{\leq_{d_{<ab>}}, \leq_{d_{<ba>}}\}$, obtained by adding $\leq_{d_{<ab>}}$ and $\leq_{d_{<ba>}}$ to L_0 and removing $\leq_q$ from L_0, is such that $\bigcap L_1 = \leq_q$ and

$$\bigvee_{\leq \in L_1} U_{(q, \leq)} = q.$$

Notice here that these "replacements" for $\leq_q$ strictly contain $\leq_q$ by the second point of Remark 2.5.3, i.e. they are "really expanding" from $\leq_q$.

If L_1 is made of only total orders we are done. Otherwise one element $\leq_{q_1}$ of L_1, where q_1 is an ultraquasi-metric, is not a total order. Hence there exist $a_1, b_1 \in X$ not comparable with $\leq_{q_1}$. By setting $d_1 := U_{(q, \leq_{q_1})} = U_{(q, \leq_{d_1})}$ (this last equality once again follows from Lemma 2.5.9), we can then repeat the same process we have done with $\leq_d$, d and points a, b to $\leq_{d_1}$, d_1 and points a_1, b_1, so that we have a collection L_2 of partial orders from "replacement". Inductively, from this process we obtain a sequence $(L_i)_{i \geq 0}$. But since X is finite and L_{i+1} is obtained from L_i by replacement of an element with "two strictly bigger" partial orders, at a certain stage, the process must end and there must be $n \geq 0$ such that we can take $L = L_n$.

(Compare with [3]). □

Remark 2.5.11.

- Obviously, the replacement of a partial order corresponds to replacement of the associated ultraquasi-metric in the supremum expression. So, in practice, since we are in a section talking of ultraquasi-metrics, it is more convenient to talk of "splitting/replacing an ultraquasi-metric" of the supremum expression instead of "replacing a partial order".
- If we have an ultraquasi-metric $d = U_{(q, \leq_d)}$ over a set X and points $a, b \in X$ which cannot be compared with $\leq_d$, then we have $u_{<ab>} = U_{(q, \leq_{u_{<ab>}})}$.

  Indeed, from inequality (2.11), we have $d_{<ab>} \leq U_{(q, \leq_{d_{<ab>}})}$ and $D_{(q, \leq_{d_{<ab>}})<ab>} \leq d_{<ab>}$. On the other hand, since $d_{<ab>}(a, b) = 0$, we have $U_{(q, \leq_{d_{<ab>}})}(a, b) = q \wedge d^{\infty}_{\leq_{d_{<ab>}}}(a, b) = 0$. Therefore, from the second bullet of Remark 2.5.2, we have $U_{(q, \leq_{d_{<ab>}})<ab>} = U_{(q, \leq_{u_{<ab>}})}$; hence the stated equality.

Corollary 2.5.10 gives us then "some way of splitting" an ultraquasi-metric defined over a finite set in such a way that the specialization order of the "splitting" ultraquasi-metrics are total orders.

Yet in general, an ultraquasi-metric obtained from the replacement process of the above corollary is not necessarily q^s-splitting.

Since the process is like "an emulation of splitting", given a finite set X and an ultrametric m defined over X, by applying to m the replacement process we described, considering the n-th step of the process, we may not choose to split/replace an m-splitting ultraquasi-metric d_n (of the supremum expression) if for any points $a_n, b_n \in X$ not comparable with $\leq_{d_n}$, the resulting "replacements" are not m-splitting.

Remark 2.5.12. Notice that for an ultrametric space (X, m) and an m-splitting ultraquasi-metric d over X that is not $U(X)$-minimally m-splitting, there exist an m-splitting ultraquasi-metric $q \leq d$ over X and points $a, b \in X$, such that we have $q(a, b) < d(a, b)$.

In addition to that, notice also that the points a, b are not comparable with $\leq_d$. Indeed, one has $m(a, b) \geq d(a, b) > q(a, b) \geq 0$; i.e. $m(a, b)$ and $d(a, b)$ are not zero. Moreover, we readily see that $m(a,b) > q(a, b)$, therefore $0 < m(a, b) = q(b, a) \leq d(b, a)$.

Hence, if we note $t = q(a, b)$, Lemma 2.5.1 gives us $q \leq u_{<ab, t>} < d$. And since q and d are m-splitting, $u_{<ab, t>}$ is m-splitting too by passing to symmetrization.

From these observations, we conclude that an ultraquasi-metric d defined over an ultrametric space (X, m) is $U(X)$-minimally m-splitting if and only if for any $a, b \in X$ not comparable with $\leq_d$ and any $l \in [0, d(a, b))$, the ultraquasi-metric $u_{<ab, l>}$ is not m-splitting.

(Compare with [3]).

Chapter 3

Main results

In this chapter we present the main results of this thesis. In particular we point out some major dif and only iferences that exist between quasi-pseudometrics and ultraquasi-pseudometrics.

3.1 Primary results

Proposition 3.1.1. *Let (X, d) be an ultraquasi-metric space. Let $a, b \in X$ with $a \neq b$. Then the ultraquasi-metric $d_{<ab>}$ is d^s-splitting if and only if $d(a, b) \leq d(b, a)$.*

Proof. First, suppose that the ultraquasi-pseudometric $d_{<ab>}$ is d^s-splitting on X. In order to reach a contradiction, assume that $d(a, b) > d(b, a)$. Then by definition of $d_{<ab>}(a, b)$, we have $d_{<a,b>} = 0$ and $d_{<ab>}(b, a) = d(b, a)$. Hence $d_{<ab>}(a, b) \neq d^s(a, b)$; which basically says that $d_{<ab>}$ cannot be d^s-splitting at (a, b)—a contraction with our supposition, thus we must have $d(a, b) \leq d(b, a)$.

In order to establish the converse, suppose now that $d(b, a) \geq d(a, b)$. We want to show that $d_{<ab>}$ is d^s-splitting. For that, assume then that there exists $(x, y) \in X \times X$ such that $d^s_{<ab>}(x, y) < d^s(x, y)$ (we recall that $d^s_{<ab>}(x, y) \leq d^s(x, y)$).

Assume then that $d^s(x, y) = d(x, y)$ and consider the four possible representations of $d^s_{<ab>}(x, y)$ according to the definition of $d_{<ab>}$. Since

$$d^s_{<ab>}(x, y) = \min(d(x, a) \vee d(b, y), d(x, y)) \vee \min(d(y, a) \vee d(b, x), d(y, x)),$$

Case 1 $d^s_{<ab>}(x, y) = (d(x, a) \vee d(b, y)) \vee (d(y, a) \vee d(b, x))$.

Case 2 $d^s_{<ab>}(x, y) = (d(x, a) \vee d(b, y)) \vee d(y, x)$.

Case 3 $d^s_{<ab>}(x, y) = d(x, y) \vee d(y, a) \vee d(b, x) = d^s(x, y) \vee d(y, a) \vee d(b, x) \geq d^s(x, y)$—Contradicting our assumption.

Case 4 $d^s_{<ab>}(x, y) = d(x, y) \vee d(y, x) = d^s(x, y)$—Contradicting our assumption.

In **Case 1**, we obtain $d(b, a) \leq d(b, x) \vee d(x, a) \leq (d(y, a) \vee d(b, x)) \vee (d(x, a) \vee d(b, y)) = d^s_{<ab>}(x, y) < d^s(x, y) = d(x, y) \leq d(x, a) \vee d(a, b) \vee d(b, y) = d(a, b)$. Which gives us then $d(b, a) < d(a, b)$, and we reach a contradiction.

In **Case 2**, we have $d(b, a) \leq d(b, y) \vee d(y, x) \vee d(x, a) = d^s_{<ab>}(x, y) < d(x, y) \leq d(x, a) \vee d(a, b) \vee d(b, y) = d(a, b)$. Then again, we have $d(b, a) < d(a, b)$—contradicting our assumption.

The case where $d^s(x, y) = d(y, x)$ is analogous.

Hence in every case we have a contradiction. Thus our initial assumption is wrong and $d_{<ab>}$ is d^s-splitting. □

Corollary 3.1.2. *Let (X, d) be an ultraquasi-metric space and let $a, b \in X$ with $a \neq b$. Then $d_{<ab>}$ or $d_{<ab>}$ is d^s-splitting.*

Proof. From Proposition 3.1.1, we notice that $d(a, b) = d(b, a)$ if and only if $d_{<ab>}$ and $d_{<ba>}$ are d^s-splitting. Then, in case of strict inequality, for example if $d(a, b) < d(b, a)$, only $d_{<ab>}$ is d^s-splitting and $d_{<ba>}$ is not. □

Proposition 3.1.3. *Let d be an ultraquasi-metric over a set X. Then d is minimal among the d^s-splitting ultraquasi-metrics defined over X if and only if the specialization order of d is a total order.*

Proof. Suppose that d is minimal among the d^s-splitting ultraquasi-metrics defined over X. In order to reach a contradiction, assume that there are $a, b \in X$ that are not $\leq_d$-comparable. We may assume that $d(a, b) \leq d(b, a)$. According to the former proposition, the ultraquasi-metric $d_{<ab>}$ is d^s-splitting over X; moreover we have $d_{<ab>} < d$ since $d_{<ab>}(a, b) = 0$ and $d(a, b) > 0$. Thus contradicting the assumption that d is $U(X)$-minimally d^s-splitting. We then conclude that $\leq_d$ is a total order.

On the other hand, suppose that the specialization order of d is a total order and assume that there exists an ultraquasi-metric h on X such that $h \leq d$ and that h is d^s-splitting. Then by our assumption about h, for any $x, y \in X$ we have $h(x, y) = 0$ in case $d(x, y) = 0$; and then $h(y, x) = d^s(x, y) = d(y, x)$ if $x > y$. Thus $h = d$—which shows that d is minimal among the d^s-splitting ultraquasi-metrics defined over X. □

Remark 3.1.4. Let (X, m) be an ultrametric space and let $\leq$ be maximal among the partial orders for which there exists an ultraquasi-metric over

X producing $(X, m, \leq)$. Then by Proposition 2.4.11, Proposition 3.1.3 and Proposition 2.4.14, such partial orders are characterized exactly as those total orders that make m Robinsonian.

Remark 3.1.5. Notice that some results implying maximally m-produced partial orders from the setting of quasi-pseudometrics become totally irrelevant because of our last observation. An example illustrating that is Corollary 3 from [3].

For an ultraquasi-metric space (X, m), the following provides an algorithm which depicts a way of constructing a strictly decreasing set of m-splitting $(d_\beta)_{\beta\leq\gamma}$ of ultraquasi-metrics. The algorithm stops as soon as an ultraquasi-metric with total order, as its specialization order, is reached. One must then consider the cases of successor ordinals and limit ordinals.

Remark 3.1.6.

(a) Let X be a set and let m be an ultra-metric defined over X. The particular steps involved in obtaining our strictly decreasing totally ordered set $(d_\beta)_{\beta\leq\gamma}$ (γ denoting the ordinal where the algorithm stops) of m-splitting ultraquasi-metrics on X is described as follow:

Set $d_0 = m$ and suppose first that β is a successor ordinal and $d_{\beta-1}$ has already been defined as an m-splitting ultraquasi-metric over X.

If the specialization order of $d_{\beta-1}$ is not a total order, then there exists $\leq_{d_{n-1}}$-incomparable elements a_β and b_β in X such that $0 < d_{\beta-1}(a_\beta, b_\beta) \leq d_{\beta-1}(b_\beta, a_\beta)$. But then by Lemma 2.5.1 and Proposition 3.1.1, respectively, we have $(d_{\beta-1})_{<a_\beta b_\beta>} < (d_{\beta-1})_{<b_\beta a_\beta>}$ and $(d_{\beta-1})_{<a_\beta b_\beta>}$ is m-splitting. Hence we define $d_\beta = (d_{\beta-1})_{<a_\beta b_\beta>}$. Suppose now that β is a limit ordinal and that the totally ordered set $(d_\alpha)_{\alpha<\beta}$ has been defined. By taking $d_\beta = \bigwedge_{\alpha<\beta} d_\alpha$ and using Proposition 2.4.1, we have an ultraquasi-metric.

(b) Let $(X, m, \leq)$ be a totally ordered ultra-metric space produced by an ultraquasi-metric e. Using the fact that $e \leq m$, we show next that our algorithm can reach e. Set $d_0 = m$. Suppose that for some successor ordinal β, for all $\alpha < \beta$, $d_\alpha \geq e$ has already been defined, but that the specialization order of $d_{\beta-1}$ is still not a total order. We are going to make a small change in the way we choose the points for the algorithm described in part (a): we choose a pair (a_β, b_β) of $\leq_{d_{\beta-1}}$-incomparable points such that $a_\beta \leq b_\beta$ (i.e. $e(a_\beta, b_\beta) = 0$). Indeed, this choice can be made since $e(a_\beta, b_\beta) = 0$ still implies that $d_{\beta-1}(a_\beta, b_\beta) \leq d_{\beta-1}(b_\beta, a_\beta)$; otherwise we would obtain $m(a_\beta, b_\beta) = d^s_{\beta-1}(a_\beta, b_\beta) = d_{\beta-1}(a_\beta, b_\beta) >$

$d_{\beta-1}(b_\beta,\, a_\beta) \geq e(b_\beta,\, a_\beta)$ and thus $e^s(a_\beta,\, b_\beta) < m(a_\beta,\, b_\beta)$ which contradicts our assumption that e is m-splitting. And from Lemma 2.5.1, we can be sure that we have $d_\alpha \geq e$ for every element $\alpha \leq \kappa$ at some stage κ where we reached a total order. Note that $d_\kappa \geq e$ and that both d_κ and e are m-splitting. Also, we have that $\leq_e = \leq = \leq_{d_\kappa}$. And finally by Remark 2.4.10, we have $d_\kappa = e$. We have then shown that our process ends at the stage κ with e.

The lemma below combined with our method from Remark 3.1.6 gives us a method of constructing more efficiently a strictly decreasing chain of ultraquasi-metrics on ultra-metric spaces $(X,\, m)$.

Lemma 3.1.7. *Let $(X,\, u)$ be an ultraquasi-metric space. Let $v : X \times X \longrightarrow \mathbb{R}$ be defined such that if $u(x,\, y) < u(y,\, x)$ we have $v(x,\, y) = 0$, and that if $u(x,\, y) \geq u(y,\, x)$ we have $v(x,\, y) = u(x,\, y)$. Then the map v is an ultraquasi-metric defined over X such that $v^s = u^s$ with $v \leq u$.*

Proof. Of course, from the definition of v we have that $v^s = u^s$ and that $v \leq u$. Indeed, if $u(x,\, y) < u(y, x)$ then we have that $v(x,\, y) = 0$ and $v(y,\, x) = u(y,\, x)$ from the second part of the definition of v.

And if $u(x,\, y) \geq u(y,\, x)$ then we have to consider two cases:

Case 1 : If $u(x,\, y) = u(y,\, x)$ then $v(x,\, y) = u(x,\, y) = u(y,\, x) = v(y,\, x)$.

Case 2 : If $u(x,\, y) > u(y,\, x)$ then $v(x,\, y) = u(x,\, y)$, and with the first part of the definition of v, we have $v(y,\, x) = 0$.

What remains to show is that v is an ultraquasi-metric over X. But since $v^s = u^s$ and since u is an ultraquasi-metric, then for $x,\, y \in X$ such that $v(x,\, y) = 0$, we have $x = y$. Hence we just need to verify the triangle inequality.

Let $x,\, y,\, z \in X$. Suppose then that we have $v(x,\, z) > v(x,\, y) \vee v(y,\, z)$.

Then necessarily, we have $0 < v(x,\, z)$. Hence by the first part of the definition of v (more accurately by considering its equivalence in contraposition), one must have $u(x,\, z) \geq u(z,\, x)$ and $v(x,\, z) = u(x,\, z)(\neq 0)$.

We are then going to evaluate the values of both $v(x,\, y)$ and $v(y,\, z)$. To do so, we then need to look at the possible values of those numbers through respectively the comparison of $u(x,\, y)$ and $u(y,\, x)$, and the comparison of $u(y,\, z)$ and $u(z,\, y)$ as implied by the way we defined v from u, above.

That is, we have four cases:

Case 1 : First, suppose that $u(x,\, y) \geq u(y,\, x)$ and $u(y,\, z) \geq u(z,\, y)$. Then we have $v(x,\, y) = u(x,\, y)$ and $v(y,\, z) = u(y,\, z)$. Thus, from what we have seen earlier and the strong triangle inequality on u, we have $v(x, z) = u(x, z) \leq u(x, y) \vee u(y, z) = v(x, y) \vee v(y, z)$, contradicting our initial assumption.

Case 2 : Next, let us suppose that we have $u(x, y) < u(y, x)$ and that $u(y, z) < u(z, y)$. Therefore, by using the strong triangle inequality on u, we have $u(x, z) \leq u(x, y) \vee u(y, z) < u(y, x) \vee u(z, y) \leq (u(y, z) \vee u(z, x)) \vee (u(z, x) \vee u(x, y)) = (u(x, y) \vee u(y, z)) \vee u(z, x) = u(z, x)$. Hence, from our initial assumption, we have $u(x, z) < u(z, x) \leq u(x, z)$: a contradiction.

Case 3 : Now, suppose that $u(x, y) \geq u(y, x)$ and that $u(y, z) < u(z, y)$. Let us consider two cases:

Case 3(a) : First, let us consider the case where $u(x, y) \geq u(y, z)$. Hence, as we have seen from our initial assumption, $v(x, z) = u(x, z) \leq u(x, y) \vee u(y, z) = u(x, y) = u(x, y) \vee 0 = v(x, y) \vee v(y, z)$, and once again a contradiction with our initial assumption.

Case 3(b) : Now, suppose that $u(x, y) < u(y, z)$. Again, from our initial assumption, we have $v(x, z) = u(x, z) \leq u(x, y) \vee u(y, z) = u(y, z) < u(z, y) \leq u(z, x) \vee u(x, y) = u(z, x) \leq u(x, z)$ which gives us another contradiction.

Case 4 : For the last case, suppose that $u(x, y) < u(y, x)$ and that $u(y, z) \geq u(z, y)$. (Even though this case is similar to that of **Case 3**, we will still go through it in detail as part of our study).

Once more, let us consider two subcases:

Case 4(a) : Suppose that $u(x, y) \leq u(y, z)$. Again, $v(x, z) = u(x, z) \leq u(x, y) \vee u(y, z) \leq u(y, z) = 0 \vee u(y, z) = v(x, y) \vee v(y, z)$ which is in contradiction with our initial assumption.

Case 4(b) : Finally, suppose that $u(x, y) > u(y, z)$. Once more, we start with $v(x, z) = u(x, z) \leq u(x, y) \vee u(y, z) = u(x, y) < u(y, x) \leq u(y, z) \vee u(z, x) = u(z, x) \leq u(x, z)$, giving us a contradiction.

Therefore our initial assumption is false; thus the strong triangle inequality holds. □

Remark 3.1.8. Notice that in Lemma 3.1.7, by simply looking at the the definition of v, we have $x \leq_v y$ if and only if $(x = y$ or $u(x, y) < u(y, x))$ for two elements $x, y \in X$. From that, we deduce $\leq_v$-incomparable elements $x, y \in X$ are characterized by $0 < u(x, y) = u(y, x)$; which as we have seen in the beginning of the proof of Lemma 3.1.7 are elements $x, y \in X$ satisfying $0 < v(x, y) = v(y, x)$. Of course the specialization order of both u and v may dif and only ifer in general. Also, we have $v(X \times X) = v^s(X \times X)$. Indeed, from that characterization of the $\leq_v$-incomparable elements, we have $v = r_{v^s} = \min(v^s, d^{\infty}_{\leq_v})$.

Example 3.1.9. Consider $X = \mathbb{R}$ with the usual order $\leq$, and define over it a map $u : X \times X \longrightarrow \mathbb{R}$ such that $u(x, y) = 0$ if $x = y$, $u(x, y) = 1$ if $x < y$, and $u(x, y) = 2$ if $x > y$. Of course, the strong triangle inequality is easily verified and we easily see that u is an ultraquasi-metric here. By applying Lemma 3.1.7 to u, we then get the ultraquasi-metric v defined such that $v(x, y) = 0$ if $x \leq y$ and $v(x, y) = 2$ otherwise.

The following example illustrates some of the concepts we have defined and seen so far about ultraquasipseudo-metrics on a finite set.

Example 3.1.10. Here, we study an example a bit similar to that of [3]. Let $X = \{a_1, a_2, a_3\}$, and define the matrix

$$\mathbf{M} = \begin{pmatrix} 0 & 4 & 5 \\ 4 & 0 & 5 \\ 5 & 5 & 0 \end{pmatrix}.$$

Then we easily check that the matrix $\mathbf{M}$ defines a ultra-metric m on X as follows,

$$m(a_i, a_j) = M_{i,j}, \quad i, j \in X.$$

1. First we illustrate the computation of $U_{(m, \leq)}$.
   Let Δ denotes the diagonal of $X \times X$. Then $\leq = \{(a_1, a_2), (a_1, a_3)\} \cup \{\Delta\}$ defines a partial order over X. That is, $\leq$ is just a partial order over X with the specification that $a_1 \leq a_2$ and $a_1 \leq a_3$. Since X is finite, we can compute $U_{(m, \leq)}$ by considering its initial definition (see (2.8) and (2.9)) and using Remark 2.4.9. Hence, the matrix which defines the ultraquasi-metric $U_{(m, \leq)}$ is given by

$$\mathbf{U} = \begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 4 \\ 5 & 5 & 0 \end{pmatrix}.$$

   And we readily see that $U^s_{(m, \leq)} = m$; so, by Corollary 2.4.4/Remark 2.4.3 and the finiteness of X, the partially ordered ultra-metric space $(X, m, \leq)$ can be produced; and the greatest of all ultraquasi-metrics producing it is $U_{(m, \leq)}$.

Let us now consider the total order $\preceq$, defined over X, such that $a_1 \prec a_2 \prec a_3$. Obviously, $\leq \subseteq \preceq$ (that is $\preceq$ extends $\leq$). Also, we can easily check that $\preceq$ makes m Robinsonian. Therefore, by Proposition 2.4.11, $(X,\, m,\, \preceq)$ can be produced by an ultraquasi-metric, and the greatest of all those quasi-metrics is $U_{(m,\,\preceq)}$. Let $\mathbf{U}'$ be the matrix representing the ultraquasi-metric $U_{(m,\,\preceq)}$. The process of computing the entries for $\mathbf{U}'$ is the same as what have been done with $\mathbf{U}$. But before doing the actual computation, since $\leq \subseteq \preceq$, we have $U_{(m,\,\preceq)} \leq U_{(m,\,\leq)}$; therefore, if $\mathbf{U}_{i,\,j} = 0$, then $\mathbf{U}'_{i,\,j} = 0$ (reducing computations). Hence,

$$\mathbf{U}' = \begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 5 & 5 & 0 \end{pmatrix}.$$

But since $\preceq$ is a total order, by Remark 2.4.10, $(X,\, m,\, \preceq)$ is uniquely produced by the ultraquasi-metric $r_m = \min(m,\, d^{\infty}_{\preceq})$. Thus, $U_{(m,\,\preceq)} = r_m = \min(m,\, d^{\infty}_{\preceq})$.

And with that, we readily note that $U_{(m,\,\preceq)}$ is $U(X)$-minimally m-splitting (highlighting Proposition 2.4.14). Indeed, by looking at the matrix $\mathbf{U}'$, if q is another ultraquasi-metric such that $q < U_{(m,\,\preceq)}$, then its matrix has to be such that all entries along the diagonal and the ones above the diagonal have to be 0; and for the entries below the diagonal, there should be an entry strictly smaller than the corresponding entry of $\mathbf{U}'$. Hence m cannot be split by q.

Notice that, every 3×3 matrix $\mathbf{S}_l$, where $0 \leq l \leq 4$, such that

$$\mathbf{S}_l = \begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & l \\ 5 & 5 & 0 \end{pmatrix},$$

defines an m-splitting ultraquasi-metric s_l such that $U_{(m,\,\preceq)} \leq s_l \leq U_{(m,\,\leq)}$.

Let us now consider the last possible total extension for $\leq$, namely

$$\sqsubseteq = \{(a_1,\, a_2),\, (a_1,\, a_3),\, (a_3,\, a_2)\} \cup \Delta.$$

Obviously, $\sqsubseteq$ is a total order. Moreover, $\sqsubseteq$ is not m-produced, i.e. there exists no ultraquasi-metric producing $(X,\, m, \sqsubseteq)$. Indeed, we have $a_1 \sqsubseteq a_3 \sqsubseteq a_2$ and $\max(m(a_1,\, a_3),\, m(a_3,\, a_2)) = \max(0,\, 5) = 5 \not\leq m(a_1,\, a_2) = 4$. Thus $\sqsubseteq$ does not make m Robinsonian. Also, if we want, the matrix of the ultraquasi-metric $U_{(m,\sqsubseteq)}$ is given by

$$\mathbf{R}' = \begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 4 \\ 4 & 0 & 0 \end{pmatrix} \quad \text{(the computation is done as with } U_{(m,\preceq)}\text{)},$$

and we readily see that $U_{(m,\sqsubseteq)}$ is not m-splitting.

2. Here we illustrate our algorithm by applying it to the initial ultrametric u.

   We start off by considering $m' := m_{<a_1a_3>}$, which gives us the matrix

$$\mathbf{M}' = \begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 4 \\ 5 & 5 & 0 \end{pmatrix}.$$

   From Lemma 3.1.7, we are supposed to compute $m'' = (m')_{<23>}$. We then obtain the associated matrix

$$\mathbf{M}'' = \begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 5 & 5 & 0 \end{pmatrix}.$$

   Then for instance we may compute $f := (m'')_{<21>}$ which has a total order as its specialization order and which is represented by the matrix

$$\mathbf{F} = \begin{pmatrix} 0 & 4 & 0 \\ 0 & 0 & 0 \\ 5 & 5 & 0 \end{pmatrix}.$$

   Note that we could have sped up the process of obtaining a total order by choosing $f = (m)'_{<21>}$.

3. In this simple example, it is indeed straightforward to give an exhaustive list of the minimally m-splitting ultraquasi-metrics on X where, of course, the matrices come in pairs since for each suitable matrix the transposed matrix determines the dual ultraquasi-metric with dual specialization order.

   (a) $\mathbf{A}_1 = U'$ with the total order $a_1 < a_2 < a_3$ (see above).

   (b) The transpose of $\mathbf{A}_1$ with its total order $a_3 < a_2 < a_1$.

   (c) $\mathbf{A}_2 = \mathbf{F}$ with its total order $a_2 < a_1 < a_3$ (see above).

(d) The transpose of $\mathbf{A}_2$ with the total order $a_3 < a_1 < a_2$.

On the other hand, the remaining two total order $a_1 < a_3 < a_2$ (see above) and $a_2 < a_3 < a_1$ are not Robinsonian and therefore cannot occur as specialization orders of m-splitting ultraquasi-metrics.

Also the matrix

$$\mathbf{B} = \begin{pmatrix} 0 & 0 & 5 \\ 4 & 0 & 5 \\ 0 & 5 & 0 \end{pmatrix},$$

and its transpose are useless, since their 0-entries do not determine an order relation on X.

3.2 Some Alternative Studies

In this section, we are going to look at results specifically applicable to join-compact ultra-metric spaces.

Proposition 3.2.1. *Let (X, d) be an ultraquasi-metric space with specialization order $\leq_d$ and define*

$$M = \{d(a', b') \mid a', b' \in X \text{ are not comparable with } \leq_d\}.$$

Assume the existence of $\max M$ *and suppose a, b non-comparable points of X such that* $\max M = d(b, a)$. *Then $d_{<ab>}$ is a d^s-splitting ultraquasi-metric.*

Proof. First, recall that $d_{<ab>}(a, b) = 0$ and $d_{<ab>}(b, a) = d(b, a)$.

If for $x \in X$ and $\beta \in [0, \infty)$, we define

$$[x]_\beta = B_{d^s}(x, \beta) = \{z \in X \mid d^s(x, z) < \beta\},$$

note that the set $\{[x]_\beta \mid x \in X\}$ is a partition of X. Indeed, for $x_1, x_2 \in X$, if there exists $x_3 \in X$ such that $d^s(x_1, x_3) < \beta$ and $d^s(x_2, x_3) < \beta$, then $d^s(x_1, x_2) < \beta$ by the strong triangle inequality. From this we deduce, once more by the strong triangle inequality, that for any $z \in X$ such that $d^s(x_1, z) < \beta$ (resp. $d^s(x_2, z) < \beta$), we have $d^s(x_2, z) < \beta$ (resp. $d^s(x_1, z) < \beta$). Hence $[x_1]_\beta = [x_2]_\beta$. If such $x_3 \in X$ does not exists, then for any $z \in X$ such that $d^s(x_1, z) < \beta$ (resp. $d^s(x_2, z) < \beta$), we must have $d^s(x_2, z) \geq \beta$ (resp. $d^s(x_1, z) \geq \beta$). Hence $[x_1]_\beta \cap [x_2]_\beta = \emptyset$. Obviously we have $\cup_{x \in X}[x]_\beta = X$.

Now, suppose that $d_{<ab>}$ is not d^s-splitting. Then there exist $x, y \in X$ such that $(d_{<ab>})^s(x, y) < d^s(x, y)$. Denote $\alpha = d^s(x, y)(\neq 0)$ so that we may write

$$(d_{<ab>})^s(x, y) < \alpha. \tag{3.1}$$

By definition of $d_{<ab>}$, we have

$$(d_{<ab>})^s(x, y) = \min(d(x, a) \vee d(b, y), d(x, y)) \vee \min(d(y, a) \vee d(b, x), d(y, x)).$$

So, the possible expressions for $(d_{<ab>})^s(x, y)$ are

Case 1 : $(d_{<ab>})^s(x, y) = d(x, a) \vee d(b, y) \vee d(y, a) \vee d(b, x)$.

Case 2 : $(d_{<ab>})^s(x, y) = d(x, a) \vee d(b, y) \vee d(y, x)$.

Case 3 : $(d_{<ab>})^s(x, y) = d(x, y) \vee d(y, a) \vee d(b, x)$.

Case 4 : $(d_{<ab>})^s(x, y) = d(x, y) \vee d(y, x) = d^s(x, y)$: we discard this one because of our assumption about x and y.

For Case 1, we have $d(b, a) \leq d(b, x) \vee d(x, a) \leq (d_{<ab>})^s(x, y)$. Similarly in Case 2, $d(b, a) \leq d(b, y) \vee d(y, x) \vee d(x, a) = (d_{<ab>})^s(x, y)$. And for Case 3, $d(b, a) \leq d(b, x) \vee d(x, y) \vee d(y, a) = (d_{<ab>})^s(x, y)$.

Therefore, by the definition of M and the pair (a, b), in all 3 cases we have

$$d^s(a, b) = d(b, a) \leq (d_{<ab>})^s(x, y) < \alpha. \tag{3.2}$$

Hence, for any of the 3 cases, we have

(i) For $z \in X$,

   (a) $d(z, a) < \alpha$ if and only if $d(z, b) < \alpha$.

   (b) $d(a, z) < \alpha$ if and only if $d(b, z) < \alpha$.

(ii) At least one of x or y does not belong to $[a]_\alpha$.

Indeed, for (i), we have $d(a, b) \leq d(b, a) < \alpha$. So, if $z \in X$ such that $d(z, a) < \alpha$ then $d(z, b) \leq d(z, a) \vee d(a, b) < \alpha$. A similar argument works for the converse and the statement (b).

For (ii), if $x, y \in [a]_\alpha$, then $d^s(a, y) < \alpha$ and $d^s(a, x) < \alpha$. Therefore, by the strong triangle inequality, $d^s(x, y) < \alpha$ which is nonsense.

Now, let us consider the 3 cases of expressions of $(d_{<ab>})^s(x, y)$ above individually.

- Case 1: from (ii), we first assume that $x \notin [a]_\alpha$. Hence $d^s(x, a) \geq \alpha$. Then, by definition of (a, b), we must have a and x comparable with $\leq_d$; otherwise we would have $d(a, x), d(x, a) \in M$, and thus $\alpha \leq d^s(a, x) \leq d(b, a)$—in contradiction with (3.2).
So, consider first the case of $x <_d a$ (strict inequality because $d^s(x, a) \geq \alpha \neq 0$). Thus $d(a, x) = d^s(a, x) \geq \alpha$. Hence, by (b), we have $d(b, x) \geq \alpha$; which gives us $d_{<ab>}(x, y) \geq \alpha$ in Case 1—in contradiction with (3.1).
On the other hand, if $x >_d a$ (strict inequality) we have $d(x, a) = d^s(x, a) \geq \alpha$, which again gives us $d_{<ab>}(x, y) \geq \alpha$ in Case 1—in contradiction with (3.1).
We still do have to consider the case of $y \notin [a]_\alpha$. But the proof is the same as the preceding one by swapping x and y in the expression of Case 1 and the $x \notin [a]_\alpha$ case, so that we end up with $d^s(y, a) \geq \alpha$ all the way through—which is then impossible too. Therefore, the Case 1 expression has to be discarded.

- Case 2: First, note that x and y can be compared with $\leq_d$ from the fact that $d^s(a, b) < \alpha = d^s(x, y)$ (by (3.2)), and the definition of (a, b). Moreover, from the expression of Case 2, we have $d(y, x) \leq (d_{<ab>})^s(x, y) < \alpha = d^s(x, y) = d(x, y)$ (this last equality results from the strict inequality in the middle). Therefore, by comparability of x, y with $\leq_d$, we have $d(y, x) = 0$, i.e. $y \leq_d x$.
Now, as with the investigations we have discussed for Case 1, from (ii), we first consider the case of $x \notin [a]_\alpha$. Hence $d^s(x, a) \geq \alpha$. Thus x and a can be compared with $\leq_d$ by definition of (a, b), since $d^s(a, b) < \alpha$ by (3.2).
Next, we then suppose that $a <_d x$, i.e. $d(x, a) = d^s(a, x) \geq \alpha$. Therefore by the expression of Case 2, $(d_{<ab>})^s(x, y) \geq \alpha$—contradicting (3.1).
Next, we suppose $x <_d a$ (strict inequality). Since we have seen that $y \leq_d x$, by transitivity of $\leq_d$, we have $y <_d a$. Thus, here, the case of $y \in [a]_\alpha$ is impossible, since then we have $d^s(a, y) < \alpha$; but we have $\alpha = d^s(x, y) = d(x, y) \leq d(x, a) \vee d(a, y) = d(a, y) = d^s(a, y) < \alpha$—a contradiction. Therefore $y \notin [a]_\alpha$, i.e. $d^s(a, y) \geq \alpha$.
Yet, we have just seen that if $y <_d a$, then $d^s(a, y) = d(a, y) \geq \alpha$. Hence, by (b), $d(b, y) \geq \alpha$. Therefore, by the expression of Case 2, $(d_{<ab>})^s(x, y) \geq \alpha$—contradicting (3.1).
Therefore we must have $x \in [a]_\alpha$ and by (ii), we then have to consider $y \notin [a]_\alpha$, i.e. $d^s(y, a) \geq \alpha$. Hence a and y can be compared with $\leq_d$ by definition of (a, b), since by (3.2) $d^s(a, b) < \alpha \leq d^s(y, a)$.

So, we then first suppose $a >_d y$. Therefore, $d^s(a, y) = d(a, y) \geq \alpha$. From this, we deduce that $d(b, y) \geq \alpha$ by (b). This then gives us $(d_{<ab>})^s(x, y) \geq \alpha$ by the expression of Case 2—resulting once more in a contradiction with (3.1).

Continuing from this observation, one must then have $a <_d y$. Since we have seen that $y \leq_d x$, then by transitivity of $\leq_d$, we have $a <_d x$. If $d(x, a) < \alpha$, then $d^s(x, y) = d(x, y) \leq d(x, a) \vee d(a, y) = d(x, a) \vee 0 < \alpha$—a contradiction. Necessarily, we must then have $d(x, a) \geq \alpha$.

Still continuing from that, since $a <_d x$, we have $d^s(x, a) = d(x, a) \geq \alpha$; which by the expression of Case 2 gives us $(d_{<ab>})^s(x, y) \geq \alpha$—in contradiction with (3.1).

Therefore, Case 2 has to be discarded, too.

- Case 3: Since the expression of Case 3 is obtained from the expression of Case 2 by swapping x and y, we obtain the proof for Case 3 by swapping x and y in the proof of Case 2. Thus Case 3 has to be discarded, too.

Therefore $d_{<ab>}$ is d^s-splitting. □

Corollary 3.2.2. *Let (X, m) be a finite ultra-metric space. If q is a $U(X)$-minimally m-splitting ultraquasi-metric, then $\leq_q$ is a total order.*

Proof. Suppose q is a $U(X)$-minimally m-splitting ultraquasi-metric and that $\leq_q$ is not a total order. Then the set

$$M = \{q(a', b') \mid a', b' \in X \text{ are not comparable with } \leq_d\} \neq \emptyset,$$

and moreover, since X is finite, there exists $(a, b) \in X \times X$ such that $\max M = q(b, a)$. Therefore, by Proposition 3.2.1, $q_{<ab>}$ is m-splitting—contradicting the minimality of q. Hence $\leq_q$ is a total order. □

The next Proposition is a generalization of Lemma 6.19 of [5].

Proposition 3.2.3. *Let (X, u) be a ultraquasi-metric space such that $(X, \tau(u^s))$ is compact. Then the set $R_0 = \{u(x, y) \mid x, y \in X$ and $x \neq y\}$ is finite or a decreasing sequence converging to 0.*

Proof. Obviously, if X is finite then R_0 is finite. Assume then X infinite and R_0 finite. Since $u^s(X \times X) \setminus \{0\} \subseteq R_0$, necessarily there exists $a_0 \in R_0$ such that the cardinality $|(u^s)^{-1}(\{a_0\})| = \infty$. By Ramsey's Theorem from graph theory we can then find a sequence $(x_n)_{n\in\mathbb{N}}$ of distinct points of X such that for any $n, m \in \mathbb{N}$ with $n \neq m$, we have $u^s(x_n, x_m) = a_0$. Therefore any extracted subsequence from that sequence will not converge, thus contradicting the compactness of $(X, \tau(u^s))$. So, if X is infinite then necessarily R_0 must be infinite.

So, assume X to be infinite. Since (X, u^s) is a compact metric space then it is separable. Hence there exists a countable dense set D in $(X, \tau(u^s))$. Let $A = \{u(x, y) \mid x, y \in D \text{ and } x \neq y\}$. Let us show $R_0 = A$. We already have $A \subseteq R_0$, let us show that $R_0 \subseteq A$. Let $x, y \in X$ such that $x \neq y$; we want to prove that there exist $a, b \in D$ such that $u(x, y) = u(a, b)$. Since D is dense, then there exist sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ of points of D such that $\lim u^s(x_n, x) = 0$ and $\lim u^s(y_n, y) = 0$. Therefore, since $u(x, y) > 0$, there exists $n_0 \in \mathbb{N}$, big enough, such that $u^s(x, x_{n_0}) < u(x, y)$ and $u^s(y, y_{n_0}) < u(x, y)$. Hence, we have $u(x, x_{n_0}) < u(x, y)$ and $u(y_{n_0}, y) < u(x, y)$. But from the strong triangle inequality we have $u(x, y) \leq u(x, x_{n_0}) \vee u(x_{n_0}, y_{n_0}) \vee u(y_{n_0}, y)$; hence $u(x, y) \leq u(x_{n_0}, y_{n_0})$. On the other hand, still from the strong triangle inequality, we have $u(x_{n_0}, y_{n_0}) \leq u(x_{n_0}, x) \vee u(x, y) \vee u(y, y_{n_0})$. Thus $u(x_{n_0}, y_{n_0}) \leq u(x, y)$. Therefore we have $u(x, y) = u(x_{n_0}, y_{n_0})$ and $R_0 = A$, i.e. R_0 is countable.

Let $(r_n)_{n\in\mathbb{N}}$ be a sequence of points of R_0. Hence there exist sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ of points of X such that for every $n \in \mathbb{N}$, we may write $r_n = u(a_n, b_n)$. On the other hand, since (X, u^s) is compact, there exist $(a, b) \in X \times X$, and a sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $\lim_{k\to\infty} u^s(a_{n_k}, a) = 0$ and $\lim_{k\to\infty} u^s(b_{n_k}, b) = 0$. But from Lemma 2.3.1, for $k \in \mathbb{N}$, we have $|u(a_{n_k}, b_{n_k}) - u(a, b)| \leq u^s(a_{n_k}, a) + u^s(b_{n_k}, b)$; so that $u(a_{n_k}, b_{n_k}) \to (a, b)$ as $k \to \infty$.

Suppose then that $(r_n)_{n\in\mathbb{N}}$ is strictly increasing. Therefore $(r_{n_k})_{n\in\mathbb{N}}$ must be strictly increasing, too; and for all $k \in \mathbb{N}$ we must have $r_{n_k} = u(a_{n_k}, b_{n_k}) < u(a, b) (\neq 0)$. Since $u(a_{n_k}, a) \vee u(b_{n_k}, b) \to 0$ and $u(a_{n_k}, b_{n_k}) \to u(a, b)$ as $k \to \infty$, at some point, there must be $k_0 \in \mathbb{N}$ such that $u(a_{n_{k_0}}, a) \vee u(b_{n_{k_0}}, b) < u(a_{n_{k_0}}, b_{n_{k_0}})$. But from this last equation and the strong triangle inequality, we get $u(a, b) \leq u(a, a_{n_{k_0}}) \vee u(a_{n_{k_0}}, b_{n_{k_0}}) \vee u(b_{n_{k_0}}, b) \leq u(a_{n_{k_0}}, b_{n_{k_0}})$—a contradiction. So, there cannot be a strictly increasing sequence in the totally ordered set $(R_0, \leq)$, where $\leq$ is the usual ordering over $\mathbb{R}$. Therefore $(R_0, \leq^*) = (R_0, \geq)$ satisfies the descending chain condition, hence it is well ordered.

Next, suppose $(r_n)_{n\in\mathbb{N}}$ is a strictly decreasing sequence. Then the subsequence $(r_{n_k})_{k\in\mathbb{N}}$ is strictly decreasing, too. Yet we know that $(r_{n_k})_{k\in\mathbb{N}}$ is a convergent subsequence of $(r_n)_{n\in\mathbb{N}}$ converging to $m(a, b)$; moreover, for any $n \in \mathbb{N}$ we can always find $k \in \mathbb{N}$ such that $r_{n_{k+1}} < r_n \leq r_{n_k}$. Hence $(r_n)_{n\in\mathbb{N}}$ converges to the same point $(r_{n_k})_{k\in\mathbb{N}}$ converges to, i.e. $(r_n)_{n\in\mathbb{N}}$ converges to $u(a, b)$.

Suppose then we have $u(a, b) > 0$. Similarly to what we have seen for the countability of R_0, one must have $u(a_{n_1}, a) \vee u(b, b_{n_1}) < u(a, b)$ for some

$n_1 \in \mathbb{N}$. Hence by the strong triangle inequality $u(a_{n_1}, b_{n_1}) \leq u(a_{n_1}, a) \vee u(a, b) \vee u(b, b_{n_1}) = u(a, b)$—contradicting the strictly decreasing property of $(r_n)_{n\in\mathbb{N}}$. Hence if $(r_n)_{n\in\mathbb{N}}$ is a strictly decreasing sequence then it must converge to zero.

As a summary for the case of R_0 being countably infinite, if we consider it as an increasing sequence $(r_n)_{n\in\mathbb{N}}$ in $(R_0, \leq)$, then at some point it becomes stationary; contradicting infiniteness of R_0. So, we may only consider it as a decreasing sequence; in fact we may assume it as a strictly descending sequence. Then in this case R_0 as a sequence converges to zero. $\square$

Theorem 3.2.4. *Let (X, u) be an ultraquasi-metric space such that $(X, \tau(u^s))$ is compact. Then there exists a $U(X)$-minimally u^s-splitting ultraquasi-metric v over X such that $v \leq u$ and $\leq_v$ is a total order.*

Proof. By Proposition 2.4.1, there exist a $U(X)$-minimally u^s-splitting ultraquasi-metric v smaller than u. Suppose then that $\leq_v$ is not a total order. Then there exist $x, y \in X$ that are $\leq_v$-incompatible. Then $v_{<x,y>} < v$ and by Proposition 3.2.1 $v_{<x,y>}$ is u^s-splitting—a contradiction. $\square$

Corollary 3.2.5. *If (X, u) is a finite ultraquasi-metric space then there exists a u^s-splitting ultraquasi-metric v over X such that $v \leq u$ and $\leq_v$ is a total order.*

Proof. By finiteness of X the topological space $(X, \tau(u^s))$ is compact. The result then just follows from the previous corollary. $\square$

Corollary 3.2.6. *If (X, u) is a compact ultra-metric space, there exists a total order $\leq$ such that $(X, u, \leq)$ can be produced.*

Proof. Again, it just follows from the previous result. $\square$

Chapter 4

Herrlich's Construction

In this chapter, we are going to present Herrlich's result [6] which is a process for obtaining a total order compatible with the topology of any ultrametric-space.

Definition 4.0.1. Let $(X, \leq)$ be a linearly (totally) ordered set. We define the *order topology* associated with $\leq$ as the topology generated by the sub-base of opens consisting of all the subsets of X of the form $]-\infty, a[$ or $]b, \infty[$ where $a, b \in X$.

A linearly ordered space or LOTS is a triple $(X, \leq, \tau(\leq))$ such that $(X, \leq)$ is a linearly ordered set and $\tau(\leq)$ is the ordered topology associated to $\leq$.

Definition 4.0.2. Using the same definition as Herrlich, a discrete order over a set is a total order such that any element has a successor and a predecessor. Moreover, the set admits a smallest and a largest element.

Lemma 4.0.3. *Any set can be discretely ordered.*

Proof. Let X be a set. In case that X is finite then we are finished. Suppose that X is not finite. By the axiom of choice, there exists a partial order $\preceq_1$ defined over X such that $(X, \preceq_1)$ is a totally ordered set. Now, consider the product set $Z = X \times \mathbb{Z}$. Let $\leq$ be the usual order over $\mathbb{Z}$ and define over Z the partial order $\preceq_2$ such that for $(x_1, z_1), (x_2, z_2) \in Z$, we have $(x_1, z_1) \preceq_2 (x_2, z_2)$ if and only if

Case 1 : $x_1 \prec_1 x_2$.

Case 2 : $x_1 = x_2$ and $z_1 \leq z_2$.

The partial order $\preceq_2$ is a total order; it is basically a lexicographic order over Z. One may consider Z as being X copies of $\mathbb{Z}$, therefore every $(x, y) \in$

$X \times \mathbb{Z}$ is located precisely in the x-th copy of $\mathbb{Z}$. The predecessor of (x, y) is of course $(x, y-1)$ whereas its successor is $(x, y+1)$. On the other hand, we have that $|Z| = |X \times \mathbb{Z}| = |X| \times |\mathbb{Z}| = |X|$ since $\mathbb{Z}$ is countable. Now, consider two new elements $-\infty$ and ∞ and let us define $L = L_0 \cup Z \cup L_1$ with $L_0 = \{-\infty\} \times \mathbb{Z}_+$ and $L_1 = \{\infty\} \times Z_-$. We define the orders on both L_0 and L_1 as the ones induced respectively by $\mathbb{Z}_+$ and $\mathbb{Z}_-$. We can then extend the total order $\preceq_2$ defined over Z to the new set L so that we have a totally ordered set $(L, \preceq_2)$. This newly formed totally ordered set is defined such that $l_0 \preceq_2 l \preceq_2 l_1$ for all $l \in Z$, $l_0 \in L_0$ and $l_1 \in L_1$. We then have $|X| = |Z| = |L|$. Therefore there is a bijection $f : X \longrightarrow L$ such that $x_1 \preceq_1 x_2$ if and only if $f(x_1) \preceq_2 f(x_2)$. Thus we have shown that X is discretely ordered. $\square$

Definition 4.0.4. Let (X, τ) be a metrizable topological space. Let $\epsilon > 0$ and let $\mathcal{U}$ be a cover of X made of clopen subsets such that each of these clopen has diameter less than ϵ. Such a cover $\mathcal{U}$ when discretely ordered is called an ϵ-chain in X.

The next result is Herrlich's contruction; proof has been given in full details.

Proposition 4.0.5. *For every ultrametric space (X, u) there exists a linear order $\leq$ such that $\tau(\leq) = \tau(u)$.*

Proof. Let $\mathcal{U}_0$ be a cover of X by open balls of radius 1. As we have seen in the proof of Proposition 3.2.1, $\mathcal{U}_0$ forms an open partition of X.

Let $\leq_0$ be a discrete ordering over $\mathcal{U}_0$.

Define by induction for $i \in \mathbb{N}$ a $1/2^i$-chain $\mathcal{U}_i$ on X and a map $f_i = (g_i, h_i)$ from $\mathcal{U}_i$ to $X \times X$ such that

1. For $U \in \mathcal{U}_i$ we have $f_i(U) \in U \times U$.
2. If $f_i(U) = (x, x)$, then $U = \{x\}$.
3. For $U \in \mathcal{U}_i$, define $\mathcal{U}_{i+1}(U)$ as a $1/2^{i+1}$-chain in U whose smallest and largest element contain respectively $g_i(U)$ and $h_i(U)$. Then define the collection
$$\mathcal{U}_{i+1} = \bigcup \{\mathcal{U}_{i+1}(U) \mid U \in \mathcal{U}_i\},$$
and defines an order $\leq_{i+1}$ over $\mathcal{U}_{i+1}$ such that
$$\leq_{i+1} = \left(\bigcup_{U \in \mathcal{U}_i} \leq_U\right) \cup \left(\bigcup_{\substack{U_1 <_i U_2 \\ U_1, U_2 \in \mathcal{U}_i}} \mathcal{U}_{i+1}(U_1) \times \mathcal{U}_{i+1}(U_2)\right).$$

That is, for $U \in \mathcal{U}_i$ and $V_1, V_2 \in \mathcal{U}_{i+1}(U)$, we have $V_1 <_{i+1} V_2$ in $\mathcal{U}_{i+1}$ if and only if $V_1 <_U V_2$ in $\mathcal{U}_{i+1}(U)$ where $\leq_U$ represents the discrete order over $\mathcal{U}_{i+1}(U)$. And for $U_1, U_2 \in \mathcal{U}_i$ with $U_1 \neq U_2$ and $V_1 \in \mathcal{U}_{i+1}(U_1)$ and $V_2 \in \mathcal{U}_{i+1}(U_2)$, we have $V_1 <_{i+1} V_2$ if and only if $U_1 <_i U_2$ in $\mathcal{U}_i$.

4. g_{i+1} is defined over $\mathcal{U}_{i+1}$ such that for $U \in \mathcal{U}_i$ if $g_i(U) \in V \in \mathcal{U}_{i+1}$, then $g_{i+1}(V) = g_i(U)$.
5. Similarly, h_{i+1} is defined over $\mathcal{U}_{i+1}$ such that for $U \in \mathcal{U}_i$ if $h_i(U) \in V \in \mathcal{V}_{i+1}$, then $h_{i+1}(V) = h_i(U)$.

Of course $\leq_{i+1}$ is a discrete order over $\mathcal{U}_{i+1}$. Let us then consider the relation

$$\leq := \bigcap_i \left(\bigcup \{U \times V \,|\, U \leq_i V \text{ for } i \in \mathbb{N}\} \right).$$

This relation is a total order over X. Indeed, let $x, y, z \in X$. Reflectivity is clear. For antisymmetry, suppose we have that $x \leq y$ and $y \leq x$. Then for every $i \in \mathbb{N}$, there exist $U_1^i, U_2^i, V_1^i, V_2^i \in \mathcal{U}_i$ such that $x \in U_1^i$, $x \in V_2^i$, $y \in U_2^i$, $y \in V_1^i$ with $U_1^i \leq_i U_2^i$ and $V_1^i \leq_i V_2^i$. Since $\mathcal{U}_i$ is a partition of X, then necessarily $U_1^i = V_2^i$ and $V_1^i = U_2^i$. Thus, since $\leq_i$ is a total order, we have $U_1^i = U_2^i = V_1^i = V_2^i$. That is, for every $i \in \mathbb{N}$, there exists $U_i \in \mathcal{U}_i$ such that $x, y \in U_i$. The sequence $(U_i)_{i\in\mathbb{N}}$ is a strictly decreasing sequence (with respect to the inclusion) of closed sets with $\operatorname{diam}(U_i) \to 0$ when $i \to 0$. Hence $x = y$.

For transitivity, suppose we have $x \leq y$ and $y \leq z$. Then for every $i \in \mathbb{N}$, there exist $U_1^i, U_2^i, V_1^i, V_2^i \in \mathcal{U}_i$ such that $x \in U_1^i$, $y \in U_2^i$, $y \in V_1^i$, $z \in V_2^i$ with $U_1^i \leq_i U_2^i$ and $V_1^i \leq_i V_2^i$. By transitivity of $\leq_i$, we have $U_1^i \leq V_2^i$. Therefore $x \leq z$ and we have the transitivity.

Of course any two elements of X can be compared with $\leq$ since every $\mathcal{U}_i$ is an ordered sequence of refining partition of X. Thus $\leq$ is a total order over X.

We then need to show that $\tau(\leq) = \tau(u)$. First, let us show that $\tau(\leq) \subseteq \tau(u)$. For each $x \in X$ and $i \in \mathbb{N}$, let $U_i(x)$ be the unique element of $\mathcal{U}_i$ containing x. Then we have $\{y \,|\, y \in X \text{ and } y < x\} = \bigcup_{i\in\mathbb{N}}\{U \,|\, U \in \mathcal{U}_i \text{ and } U <_i U_i(x)\}$, i.e. can be expressed as union of elements of $\tau(u)$. Similarly $\{y \,|\, y \in X \text{ and } x < y\} \in \tau(u)$. Thus $\tau(\leq) \subseteq \tau(u)$.

For the other inclusion, $\bigcup_{i\in\mathbb{N}} \mathcal{U}_i$ is a basis for $\tau(u)$. So, to show that $\tau(u) \subseteq \tau(\leq)$, we need to show that for each $i \in \mathbb{N}$ and $U \in \mathcal{U}_i$ we have $U \in \tau(\leq)$. Hence, let $U \in \mathcal{U}_i$. We can then assume that there exist $U_0, U_1 \in \mathcal{U}_i$ such that $U_0 \leq_i U \leq_i U_1$ with U_0 and U_1 being respectively the direct predecessor and direct successor of U. Thus we have $U = \{x \mid h_n(U_0) < x \text{ and } x < g_n(U_1)\}$. Thus $U \in \tau(\leq)$. The cases where U is the smallest or the largest element of $\mathcal{U}_i$ are similar. Hence $U \in \tau(\leq)$ in all cases and $\tau(u) \subseteq \tau(\leq)$. $\square$

Chapter 5

Conclusion

As we have seen, the major result of the thesis is Proposition 3.1.3 which helps us characterize the $U(X)$-minimally splitting ultraquasi-metrics as those that have a total order as their specialization order.

We also have generalized Lemma 6.19 from [5] by using joincompact (or supcompact) ultraquasi-metric space instead of compact ultraquasi-metric space.

We also have discussed both the bicompletion and the Herrlich's construction.

In the beginning, we noted that an ultraquasi-metric is a quasi-metric, a natural question that may arise is then when does the $U(X)$-minimally m-splitting ultraquasi-metric element from Proposition 2.4.1 and the equivalent minimally m-splitting quasi-metric element of that same space coincide. Of course there will be many minimal elements of either kind in general.

Also, some questions still arise concerning Herrlich's construction. Indeed, we see that we have countably many levels (partitions). A natural thing to do then is to ask if we can generalize the construction by overcoming the limitations of countability with ordinals.

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